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14. ABSTRACT

Propagation (optical) and scattering (radar) transformations using Jones and Sinclair matrices, accordingly, have been presented in both matrix and quaternionic forms exhibiting full agreement of the two approaches. The Kennaugh's inversion point techniques has been adopted to the Jones and nonsymmetrical Sinclair matrix. Poincaré sphere models of those matrices have been presented and compared. Both spheres have same diameter and common inversion point but different axes and angles of rotation. Those differences have been explained by the dependence of description of the emerging wave's polarization on reversal of the propagation *z*-axis on the output.

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Optical and bistatic radar polarimetry. Tangential polarization phasors. Inversion point techniques. Time and spatial frame reversal. Quaternions of polarization vectors and scattering matrices.

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FINAL REPORT

TO THE OFFICE OF NAVAL RESEARCH

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General Remarks about the Final Report

This is an updated and enlarged version on my ONR-Report-6 sent to Professor Pellat-Finet and the Program Coordinator on October 11, 2002. That 6th report presented, for the first time, my amplitude matrix approach translated into quaternionic form. My matrix approach has been concisely described in the earlier elaborated the ONR-Report-5 (60 pages text), dated 31 July 2002. Its comparatively short form had to be more suitable for publication in the capacity allowed. Full text of 240 pages has been presented in my ONR-Report-3 (Final Version) dated 1 June 2001. That was the Final Report of the previous grant: N00014-00-1-0620 (PR # 00PR06427-00 and 01PR07224-00).

It is believed that the text of this Final Report fully adopts the Kennaugh's inversion point techniques to the general coherent bistatic scatter case. Different representations of 'spinorial scatterers', of the Jones and Sinclair type, on the Poincaré sphere equipped with the Kennaugh's inversion point could be demonstrated owing to use of the time and spatial frame reversal concepts. The use of quaternions (proposed by Pellat-Finet) was possible taking advantage of their close relation to Pauli matrices by which amplitude matrices of scatterers can be expressed. To stress that link between amplitude matrices and their quaternionic forms the same symbols have been used for them with the only difference in the type of fonts: the 'Times New Roman' or 'Lucida Handwritting'. Therefore, practically, direct translation of the matrix to quaternionic formulae was possible leading to the end result in the form strictly corresponding to those with the Mueller or Kennaugh matrices.

Zbigniew H. Czyż

Zbigniew H. Czyż

Warsaw, December 7, 2002

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FOREWORD

Fundamental question of the polarimetry theory in communications and radar is how to describe polarization of antennas and waves, the process of polarization transformation when scattering, and how to express the received voltage with its phase in terms of polarization parameters of transmitting and receiving antennas and those of scattering object.

Only the first two questions have been answered by researchers in optics. It has been established that the polarization and phase of the time-harmonic locally plane waves propagating in a vacuum depend on three real parameters. Two of them describing polarization can be presented by a point on the Poincaré sphere, while eight real parameters of the scattering object sufficiently determine transformation of polarization and phase when scattering ([1], [5]).

The problem appeared more complex in the case of partial polarization of waves treated as the incoherent sum of those completely polarized. Introduction of the fourth real parameter appeared necessary together with 16 real parameters of the scattering object. As an example of such presentation can serve Stokes 4-vectors and Mueller matrices being used in a statistical optics approach ([1], [2]).

Next problem arose in communications and radar considering waves propagating in opposite directions. The question was, for example, how to identify three real parameters of each antenna employed in the direct transmission independently of its destination as transmit or receive one, and how to use those six parameters to express amplitude and phase of the received voltage. One of satisfactory solutions to this problem has been presented in this text by introduction of operation of the local spatial frame reversal by its rotation by 180^0 about an axis perpendicular to the propagation axis. Of course, such approach was possible when relating polarization to the spatial frame. That can be considered as a disadvantage of the method but the results obtained, evidence of description of the problem, seem to justify such an approach.

Having main idea, mathematical tools have to be chosen leading to the most simple presentation of polarimetric operations. Those tools can be divided into two categories: analytical and geometrical. Geometrically, the elliptically polarized wave can be represented by a polarization helix, here proposed and better serving to define the complete polarization than the polarization ellipse. Kennaugh [6] extended the Poincaré sphere concept to present geometrical models of scattering operators by introduction the concept of the inversion point located inside the sphere. That concept has been further extended to broader class of scattering operators (originally applied to backscattering) what has been reported also in this text.

There are many ways of analytical treatment of optical and radar polarimetry. Interesting reviews have been presented by many authors dealing with the radar polarimetry theory. In opinion of this author, most convenient for direct computations is the polarization and phase column vectors and scattering matrices approach. However, there exist an approach which closely links polarimetry with relativistic kinematics described by Pellat-Finet [10]. That is the quaternion approach which, what should be stressed, is closely related to the matrix approach (as well as to the spinorial approach [7]) and will be also carefully presented in this text. Both matrix and quaternionic transformations have been interpreted geometrically by using models of scattering operators in the form of the Poincaré sphere additionally equipped with the inversion point. Also, polarization phasors tangent to the sphere at the polarization points appeared necessary to precisely determine the bases of polarization and phase vectors or quaternions.

1. AN INTRODUCTION. THE SPHERE OF TANGENTIAL POLARIZATION PHASORS

The beneath conducted considerations lead to construction of the Poincare sphere models of the Jones (Section 2) and Sinclair (Section 3) matrices, or their corresponding quaternions, presented in any orthogonal elliptical basis determined by one tangential polarization (TP) phasor B. Its orthogonal mate, the collinear $B\text{x}$ phasor of the same phase, will be uniquely determined.

The choice of some special bases can be convenient because it may lead to simpler forms of transformation matrices/quaternions and symmetrical locations of special polarization points on the Poincare sphere, properly rotated versus the original linear polarization basis. Also, decomposition of the Jones and Sinclair matrices into product of matrices representing real and imaginary rotation of the illuminating polarization and phase (PP) vector, corresponding to the TP phasor and valid in any such a basis, enables one to indicate intrinsic properties of those polarimetric operators. Therefore, the concept of the polarization sphere of the TP phasors and its phasor basis should be well understood.

The TP phasor sphere differs from the known Poincaré sphere (of polarization points) by introduction of additional information about the phase. With respect to the polarization and phase E_{2c} space of the PP vectors (compare [8]), the TP phasor sphere can be considered as its 'squared space' with all three angles, determining the complete polarization and phase, doubled.

The Poincare sphere and the sphere of the TP phasors are both related to the spatial frame which usually is the local right-handed *xyz* coordinate system with the *z* coordinate chosen along the propagation axis. The equator of linear polarization points on the Poincare sphere of unit radius corresponds to one half of the unit circle in the *xy* plane. On the sphere of the TP phasors one deals with the 'double equator', of the angular length of 4π, corresponding to the full circle in the *xy* plane and resembling the Moebius tape, the one-sided surface. The TP phasor oriented and shifted along that tape comes to the same location at the tape after the complete 4π shift. Its phase doesn't change when shifting along the tape, the polarization changes only. After 2π shift it arrives at the 'opposite polarization' point but of the same phase. It arrives at the orthogonal polarization point after the first π shift.

Each small circle of that sphere is also of the 4π lengths! However, when shifted along such a circle, not only the polarization but also the phase is changing. The sum of the double change of polarization (equal to the solid angle subtended by the circle, in radians) plus the double change of phase (the effective rotation of the phasor on its path) equals 4π. When the small circle reduces to a point, the phase is only that what changes and the 4π rotation represents the 2π change of phase.

Location and orientation of any phasor P of the elliptical polarization (tangent to the Poincare sphere at the polarization point P) can be determined in relation to another phasor, for example the original basis phasor X of linear polarization and null phase (tangent to the sphere at the point X and oriented along the equator of linear polarizations). The X phasor represents the x direction of polarization. It becomes the P phasor after:

- rotation about the OX axis (called also the Q_X axis) of the sphere by the $2\delta_X^P$ angle of the right-hand-sense (denoting the double phase delay for the angle of positive value), then
- shifting in direction of its arrow to the point P by the angle of $2\gamma_X^P$ (the double angular distance between polarizations represented by the X and P points), and
- the $2\varepsilon_X^P$ rotation about the OP axis (it is another double phase delay).

Ranges of those angles will be chosen as follows:

$$0 \le 2\gamma \le \pi, \qquad -\pi < 2\delta \le \pi, \qquad -2\pi \le 2\varepsilon \le 2\pi \tag{1.1}$$

with $2\delta = 0$ for $2\gamma = 0$ and $2\gamma = \pi$. The couple of 2γ and 2δ angles will uniquely determine any polarization point on the Poincare sphere. Two identically oriented phasors tangent at the same point but of 2ε parameter differing by 2π will be considered as representing the same polarization and the opposite phase. The alternative interpretation: of the same phase but of 'opposite polarization' will be excluded from considerations by the above proposed ranges of angular parameters.

Any basis phasor B will be determined by other three angles, $2\gamma_X^B, 2\delta_X^B, 2\varepsilon_X^B$. The second phasor of that basis, Bx, will be uniquely determined by shifting B to its antipodal point on the Poincare sphere in direction indicated by the B phasor's arrow. The change of the TP basis $X \rightarrow B$ results also in the corresponding change of the Stokes parameter (right-

handed rectangular) coordinate system $\mathrm{Q}_X\mathrm{U}_X\mathrm{V}_X \to \mathrm{Q}_B\mathrm{U}_B\mathrm{V}_B$. For any *B* basis, the Q_B axis will cross the Poincare sphere at the B point and the new equator $\mathrm{V}_B = 0$, as well as the U_B axis, will be indicated by the arrow of the *B* phasor.

The unit column polarization and phase (PP) vector in the *X* basis will be of the form

$$u_X^P = \begin{bmatrix} \cos\gamma \exp[-j(\delta+\varepsilon)] \\ \sin\gamma \exp[+j(\delta-\varepsilon)] \end{bmatrix}_X^P \equiv \begin{bmatrix} a \\ b \end{bmatrix}_X^P \tag{1.2}$$

The change of the TP phasor basis $X \to B$ for the column PP vector can be expressed by the unitary matrix transformation (see Appendix A, formula (A.2)):

$$\begin{aligned} u_X^P \to u_B^P &= C_B^X u_X^P \equiv \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}_B^X u_X^P \\ &= \widetilde{C}_X^B{}^* u_X^P = \begin{bmatrix} a^* & b^* \\ -b & a \end{bmatrix}_X^B u_X^P; \quad \det C_X^B = +1 \end{aligned} \tag{1.3}$$

The corresponding column Stokes four-vector depends on the double angular parameters 2γ and 2δ:

$$\mathbf{P}_X^P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \cos 2\gamma \\ \sin 2\gamma \cos 2\delta \\ \sin 2\gamma \sin 2\delta \end{bmatrix}_X^P = \frac{1}{\sqrt{2}} \begin{bmatrix} aa^* + bb^* \\ aa^* - bb^* \\ ab^* + ba^* \\ j(ab^* - ba^*) \end{bmatrix}_X^P \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \mathrm{q} \\ \mathrm{u} \\ \mathrm{v} \end{bmatrix}_X^P \tag{1.4}$$

Similar change of the TP phasor basis can be expressed by the following orthogonal matrix transformation equation (see Appendix E, formula (E.2))

$$\mathbf{P}_X^P \to \mathbf{P}_B^P = \mathbf{D}_B^X \mathbf{P}_X^P = \widetilde{\mathbf{D}}_X^B \mathbf{P}_X^P; \quad \mathbf{D}_B^X \widetilde{\mathbf{D}}_B^X = \mathbf{D}_B^X \mathbf{D}_X^B = diag\{1,1,1,1\} \tag{1.5}$$

with the real 4x4 change-of-basis matrix

$$D_X^B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\gamma & -\sin 2\gamma \cos 2\varepsilon & \sin 2\gamma \sin 2\varepsilon \\ 0 & \sin 2\gamma \cos 2\delta & \cos 2\gamma \cos 2\delta \cos 2\varepsilon - \sin 2\delta \sin 2\varepsilon & -\cos 2\gamma \cos 2\delta \sin 2\varepsilon - \sin 2\delta \cos 2\varepsilon \\ 0 & \sin 2\gamma \sin 2\delta & \cos 2\gamma \sin 2\delta \cos 2\varepsilon + \cos 2\delta \sin 2\varepsilon & -\cos 2\gamma \sin 2\delta \sin 2\varepsilon + \cos 2\delta \cos 2\varepsilon \end{bmatrix}_X^B$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{p} & \mathbf{q}_R & \mathbf{q}_I \end{bmatrix}_X^B \tag{1.6}$$

with

$$|\mathbf{p}|=|\mathbf{q}_R|=|\mathbf{q}_I|=1,$$
$$\mathbf{p}\bullet\mathbf{q}_R=\mathbf{q}_R\bullet\mathbf{q}_I=\mathbf{q}_I\bullet\mathbf{p}=0, \tag{1.7}$$
$$\mathbf{p}\times\mathbf{q}_R=\mathbf{q}_I,\quad \mathbf{q}_R\times\mathbf{q}_I=\mathbf{p},\quad \mathbf{q}_I\times\mathbf{p}=\mathbf{q}_R.$$

and where $\mathbf{q}_R$ is a vector tangent to the sphere at the point B and its orientation is exactly that of the B phasor, or that of the $-B$ phasor which differs from B by the angle $2\varepsilon=2\pi$.

What should be observed is that the orthogonal transformation matrix depends on all three double angular parameters, though the four-vector itself bears no information about the phase. That is the result of rotation of the Poincare sphere with the TP phasors of the PP basis by three Euler angles $2\varepsilon_X^B, 2\gamma_X^B, 2\delta_X^B$ or by one angle $2\phi_X^B$ about a determined $n_X^{X\to B}$ axis of any orientation (for details see again Appendix A, substituting phasors X for P and B for K). For that reason to precisely describe the orthogonal elliptical PP basis it is insufficient to indicate the orthogonal polarizations only, without phases of the PP basis vectors determined by directions of basis phasors.

2. OPTICAL POLARIMETRY. THE PROPAGATION TRANSFORMATION

2.1 The Jones matrix decomposition into product of matrices

The time-harmonic electromagnetic polarimetry considers plane waves propagating in a homogeneous isotropic lossless medium in which they can be sufficiently described by electric vectors only because their magnetic mates are unambiguously determined. Those waves meet obstacles on their path changing their amplitudes polarizations and phases being subjects to determination.

Polarization is always described versus an established local spatial frame, a right-handed xyz coordinate system with the z axis chosen along the propagation path. In optical polarimetry z axes of those local spatial frames are always directed to the obstacle for the incoming wave and out of the obstacle for the outgoing (emerging) wave. So, all waves under consideration propagate in the $+z$ directions only. Electric vectors of those waves, incoming and outgoing, have two components, along x and y axes. Those components used to be considered as complex quantities because otherwise it would be impossible to separate the three interesting terms: the real amplitude, E_0, polarization and phase (PP) unit complex vector, u_X^T, and the wave factor, $\exp[j(\omega t-kz)]$. Therefore, the real components are treated

as real parts of complex numbers. That leads to the following expressions for electric column vector waves:

$$
{}^{+}\begin{bmatrix} E_x \\ E_y \end{bmatrix}^T (t,z) = \mathrm{Re}\{E_0^T u_X^T \exp[j(\omega t - kz)]\} \tag{2.1}
$$

Only the PP vectors are subject to transformation by obstacles. An expression of such transformation takes the form of an equation with the amplitude 2x2 complex Jones matrix (see [1], [2], [5])

$$
A_X^{\circ} u_X^T = \lambda^T u_X^{So} \tag{2.2}
$$

Here X, T, So are tangential phasors: of the original linear polarization basis and of the incoming and outgoing waves, accordingly.

Generally, the (amplitude) Jones matrix A_B°, in any B basis, will be presented with two indexes: the lower, denoting the basis, and the upper, indicating its dependence on the Sinclair matrix A_B through the output spatial frame reversal-by-rotation matrix C_B°:

$$
A_B^{\circ} = C_B^{\circ} * A_B \tag{2.3}
$$

(see Section 3 for the detailed explanation).

The following decomposition of the Jones matrix

$$
A_B^{\circ} = \begin{bmatrix} A_2^{\circ} & A_3^{\circ} \\ A_4^{\circ} & A_1^{\circ} \end{bmatrix}_B = e^{j\xi^{\circ}} \frac{\sqrt{\sigma_0}}{2} C_B^{ROT^{\circ}} A_{nB}^{LOR} \tag{2.4}
$$

can be checked, after rather tedious work, by direct substitution of the following rotation and Lorentz matrices (see also Appendix A, formulae (A.3) - (A.7), justifying the last form of the rotation matrix):

$$
\begin{aligned}
C_B^{ROT^{\circ}} &= \frac{1}{\sqrt{\sigma_0}} \left\{ \left(A_B^{\circ} e^{-j\xi^{\circ}} \right) + C^{\times} \left(A_B^{\circ} * e^{j\xi^{\circ}} \right) \widetilde{C}^{\times} \right\} \\
&= \frac{1}{\sqrt{\sigma_0}} \begin{bmatrix} A_2^{\circ} e^{-j\xi^{\circ}} + A_1^{\circ} * e^{j\xi^{\circ}} & A_3^{\circ} e^{-j\xi^{\circ}} - A_4^{\circ} * e^{j\xi^{\circ}} \\ A_4^{\circ} e^{-j\xi^{\circ}} - A_3^{\circ} * e^{j\xi^{\circ}} & A_1^{\circ} e^{-j\xi^{\circ}} + A_2^{\circ} * e^{j\xi^{\circ}} \end{bmatrix}_B \\
&= \begin{bmatrix} \cos\phi^{\circ} - jn_1^{\circ} \sin\phi^{\circ} & (-n_3^{\circ} - jn_2^{\circ}) \sin\phi^{\circ} \\ (n_3^{\circ} - jn_2^{\circ}) \sin\phi^{\circ} & \cos\phi^{\circ} + jn_1^{\circ} \sin\phi^{\circ} \end{bmatrix}_B
\end{aligned} \tag{2.5}
$$

with

$$\begin{bmatrix} \cos\phi^\circ \\ n_1^\circ \sin\phi^\circ \\ n_2^\circ \sin\phi^\circ \\ n_3^\circ \sin\phi^\circ \end{bmatrix}_B = \frac{1}{\sqrt{\sigma_0}} \begin{bmatrix} \mathrm{Re}\{(A_2^\circ + A_1^\circ)\exp(-j\xi^\circ)\} \\ -\mathrm{Im}\{(A_2^\circ - A_1^\circ)\exp(-j\xi^\circ)\} \\ -\mathrm{Im}\{(A_4^\circ + A_3^\circ)\exp(-j\xi^\circ)\} \\ \mathrm{Re}\{(A_4^\circ - A_3^\circ)\exp(-j\xi^\circ)\} \end{bmatrix}_B \tag{2.6}$$

and

$$\begin{aligned} A_{nB}^{LOR} &= e^{-j\xi^0} \frac{2}{\sqrt{\sigma_0}} \widetilde{C}_B^{ROT^\circ} * A_B^\circ \\ &= \frac{2}{\sigma_0} \left\{ \widetilde{A}_B^\circ + C^\times \widetilde{A}_B^\circ \widetilde{C}^\times e^{-j2\xi^0} \right\} A_B^\circ \\ &= \frac{2}{\sigma_0} \begin{bmatrix} M_2^\circ + M_4^\circ + |\det A| & A_3^\circ A_2^\circ * + A_1^\circ A_4^\circ * \\ A_2^\circ A_3^\circ * + A_4^\circ A_1^\circ * & M_3^\circ + M_1^\circ + |\det A| \end{bmatrix}_B \\ &= \begin{bmatrix} 1 - \mathrm{Q} & -\mathrm{U} + j\mathrm{V} \\ -\mathrm{U} - j\mathrm{V} & 1 + \mathrm{Q} \end{bmatrix}_{nB}^{\mathrm{I}} \end{aligned} \tag{2.7}$$

where $\mathrm{Q}_{nB}^{\mathrm{I}}, \mathrm{U}_{nB}^{\mathrm{I}}$ and $\mathrm{V}_{nB}^{\mathrm{I}}$ can be considered as coordinates of the inversion point I inside the Poincare sphere of unit radius in the B basis and are expressed by elements of the first raw of the Mueller (and Kennaugh) matrix as follows (see Appendix E):

$$\begin{bmatrix} \mathrm{Q} \\ \mathrm{U} \\ \mathrm{V} \end{bmatrix}_{nB}^{\mathrm{I}} = \frac{-2}{\sigma_0} \begin{bmatrix} \mathrm{b}_1 \\ \mathrm{b}_3 \\ \mathrm{b}_5 \end{bmatrix}_B ; \text{ with } \begin{array}{l} \mathrm{b}_1 = \tfrac{1}{2}(\mathrm{M}_{2B}^\circ - \mathrm{M}_{3B}^\circ + \mathrm{M}_{4B}^\circ - \mathrm{M}_{1B}^\circ); \;\; \mathrm{M}_{kB}^\circ = A_{kB}^\circ A_{kB}^\circ *, \;\; k = 1,2,3,4 \\ \mathrm{b}_3 + j\mathrm{b}_5 = A_{2B}^\circ A_{3B}^\circ * + A_{4B}^\circ A_{1B}^\circ * \end{array} \tag{2.8}$$

and

$$\sigma_0 = M_{2B}^\circ + M_{3B}^\circ + M_{4B}^\circ + M_{1B}^\circ + 2|\det A_B^\circ| \tag{2.9}$$

Coordinates $\mathrm{Q}_{nB}^{\mathrm{I}}, \mathrm{U}_{nB}^{\mathrm{I}}$ and $\mathrm{V}_{nB}^{\mathrm{I}}$ are called the normalized Stokes parameters of that point.

There is another way to decompose the Jones matrix expressed in any orthogonal elliptical basis B, by a matrix C_B of the 'complex pure retarder' of unit determinant multiplied by a coefficient (complex number) as defined, for example, in Pellat-Finet [10], Optik, 1992, p.106:

$$A_B^\circ = \sqrt{\mathrm{a}_0} e^{j\xi^\circ} C_B \tag{2.10}$$

with

$$\begin{aligned} \mathrm{a}_0 &\equiv |\det A_B^{\circ}| \\ \xi^{\circ} &= \arg A_B^{\circ} = \tfrac{1}{2}\arg\det A_B^{\circ}; \qquad -\pi \le \xi^{\circ} \le \pi \end{aligned} \tag{2.11}$$

However, in general case, the matrix cannot be presented in such a form. This is because determinants of some A_B° matrices may equal zero and then the whole matrix vanishes identically. Fortunately, there is a way to omit that difficulty. Splitting the retarder matrix into product of pure and hyperbolic rotation matrices (the last can be called also the amplitude boost matrix),

$$C_B = C_B^{ROT^{\circ}} C_B^{BST}; \qquad \det C_B^{ROT^{\circ}} = \det C_B^{BST} = 1 \tag{2.12}$$

with

$$C_B^{BST} = \begin{bmatrix} \cosh\alpha + m_{1B}\sinh\alpha & (m_{2B} - jm_{3B})\sinh\alpha \\ (m_{2B} + jm_{3B})\sinh\alpha & \cosh\alpha - m_{1B}\sinh\alpha \end{bmatrix}_B \tag{2.13}$$

one can introduce, instead of C_B^{BST}, a new amplitude matrix, A_B^{LOR}, owing to which the Jones matrix will not vanish identically with a_0. Defining:

$$\begin{aligned} A_B^{LOR} &= \sqrt{\mathrm{a}_0}\, C_B^{BST} \equiv \frac{\sqrt{\sigma_0}}{2}\frac{1}{\cosh\alpha} C_B^{BST} \\ &= \frac{\sqrt{\sigma_0}}{2}\begin{bmatrix} 1 + m_{1B}\tanh\alpha & (m_{2B} - jm_{3B})\tanh\alpha \\ (m_{2B} + jm_{3B})\tanh\alpha & 1 - m_{1B}\tanh\alpha \end{bmatrix}_B \end{aligned} \tag{2.14}$$

one obtains

$$\begin{aligned} \det A_B^{LOR} &= \frac{\sigma_0}{4}\frac{1}{\cosh^2\alpha} = \frac{\sigma_0}{4}(1 - \tanh^2\alpha) = \mathrm{a}_0 \\ \tfrac{1}{2} Span A_B^{LOR} &= \frac{\sigma_0}{4}(1 + \tanh^2\alpha) = \tfrac{1}{2} Span A_B \equiv \mathrm{a}_1 \end{aligned} \tag{2.15}$$

with

$$\begin{aligned} \sigma_0 &= Span A_B^{LOR} + 2\det A_B^{LOR} \\ &= 2(\mathrm{a}_1 + \mathrm{a}_0) = Span A_B + 2|\det A_B| \end{aligned} \tag{2.16}$$

and

$$A_{nB}^{LOR} = \frac{2}{\sqrt{\sigma_0}} A_B^{LOR} \tag{2.17}$$

Finally, the Jones matrix can be presented in the originally proposed form

$$A_B^{\circ} = e^{j\xi^{\circ}} C_B^{ROT^{\circ}} A_B^{LOR} = e^{j\xi^{\circ}} \frac{\sqrt{\sigma_0}}{2} C_B^{ROT^{\circ}} A_{nB}^{LOR} \tag{2.18}$$

In such a form the Jones matrix for $a_0 = |\det A_B^{\circ}| = 0$ will not vanish identically because its coefficient $\sqrt{a_0}$ has been exchanged for $\sqrt{\sigma_0}/2$ which is always greater than zero.

Further (see Section 2.7), $r_0 \equiv \sqrt{\sigma_0}/2$ and $r_{0n} \equiv \sqrt{\sigma_{0n}}/2 = 1$ will be interpreted geometrically as radii of the Poincare sphere models, regular and normalized, of the Jones or Lorentz matrices.

2.2 Quaternionic forms of the Lorentz and rotation-after-Lorentz amplitude transformation matrices

Quaternionic expression for the Jones matrix can be determined by introduction of the following mutual dependences between Pauli matrices and their quaternionic versors (see Pellat-Finet [10], Optik, 1992, p.101, or Misner et al [7], Chapter 41, p.1136):

$$\underline{\sigma}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Leftrightarrow e_0 \equiv 1, \quad \underline{\sigma}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Leftrightarrow je_1, \quad \underline{\sigma}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Leftrightarrow je_2, \quad \underline{\sigma}_3 = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} \Leftrightarrow je_3 \tag{2.19}$$

Those versors undergo multiplication rules:

$$e_k^2 = -1, \quad k = 1,2,3; \quad e_1 e_2 = e_3 = -e_2 e_1, \quad e_2 e_3 = e_1 = -e_3 e_2, \quad e_3 e_1 = e_2 = -e_1 e_3 \tag{2.20}$$

In terms of those versors the quaternionic form of the Lorentz amplitude matrix can be found by applying the following procedure employing formerly established dependence between the inversion point I coordinates and components of the $\underline{m}_B$ axis of the imaginary rotation (compare the last formula of (2.7), and (2.14) with (2.17)):

$$\begin{aligned} A_{nB}^{LOR} &= \begin{bmatrix} 1-\mathrm{Q} & -\mathrm{U}+j\mathrm{V} \\ -\mathrm{U}-j\mathrm{V} & 1+\mathrm{Q} \end{bmatrix}_{nB}^{\mathrm{I}} \\ &= \underline{\sigma}_0 - (\mathrm{Q}\underline{\sigma}_1 + \mathrm{U}\underline{\sigma}_2 + \mathrm{V}\underline{\sigma}_3)_{nB}^{\mathrm{I}} \\ &= \underline{\sigma}_0 + (m_1\underline{\sigma}_1 + m_2\underline{\sigma}_2 + m_3\underline{\sigma}_3)_B \tanh\alpha \end{aligned} \tag{2.21}$$

$$
\begin{aligned}
&= \frac{1}{\cosh\alpha}\left\{\cosh\alpha\underline{\sigma}_0 + (m_1\underline{\sigma}_1 + m_2\underline{\sigma}_2 + m_3\underline{\sigma}_3)_B \sinh\alpha\right\} \\
&= \frac{1}{\cosh\alpha}\left\{\cos j\alpha\underline{\sigma}_0 - j(m_1\underline{\sigma}_1 + m_2\underline{\sigma}_2 + m_3\underline{\sigma}_3)_B \sin j\alpha\right\} \\
&= \frac{1}{\cosh\alpha}\exp(-j\underline{m}_B j\alpha) = \frac{1}{\cosh\alpha}\exp(\underline{m}_B\alpha) \Leftrightarrow \frac{1}{\cosh\alpha}\exp(m_B j\alpha)
\end{aligned}
\tag{2.22}
$$

with the matrix

$$
\underline{m}_B = (m_1\underline{\sigma}_1 + m_2\underline{\sigma}_2 + m_3\underline{\sigma}_3)_B \equiv \frac{-1}{\tanh\alpha}(\mathrm{Q}\underline{\sigma}_1 + \mathrm{U}\underline{\sigma}_2 + \mathrm{V}\underline{\sigma}_3)^{\mathrm{I}}_{nB} = (\mathrm{q}\underline{\sigma}_1 + \mathrm{u}\underline{\sigma}_2 + \mathrm{v}\underline{\sigma}_3)^{\mathrm{M}}_B \Leftrightarrow jm_B
\tag{2.23}
$$

The $\mathcal{A}^{LOR}_{nB}$ quaternion corresponds to its matrix form as follows:

$$
\begin{aligned}
\mathcal{A}^{LOR}_{nB} &= 1 - j(\mathrm{Q}e_1 + \mathrm{U}e_2 + \mathrm{V}e_3)^{\mathrm{I}}_{nB} \\
&= 1 + j(m_1e_1 + m_2e_2 + m_3e_3)_B \tanh\alpha \\
&= \frac{1}{\cosh\alpha}\exp(m_B j\alpha)
\end{aligned}
\tag{2.24}
$$

with the (pure) quaternion m_B representing the axis of a hyperbolic (imaginary) rotation in the B basis

$$
m_B = (m_1e_1 + m_2e_2 + m_3e_3)_B \equiv (\mathrm{q}e_1 + \mathrm{u}e_2 + \mathrm{v}e_3)^{\mathrm{M}}_B
\tag{2.25}
$$

with

$$
\mathrm{q}^2 + \mathrm{u}^2 + \mathrm{v}^2 = 1
\tag{2.26}
$$

Those normalized Stokes parameters, coordinates of the polarization points on the Poincare sphere of unit radius, can be expressed in terms of coordinates of the M point representing polarization of the incident wave corresponding to the maximum emerging power:

$$
\begin{bmatrix} \mathrm{q} \\ \mathrm{u} \\ \mathrm{v} \end{bmatrix}^{\mathrm{M}}_B = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}_B = \frac{-1}{\tanh\alpha}\begin{bmatrix} \mathrm{Q} \\ \mathrm{U} \\ \mathrm{V} \end{bmatrix}^{\mathrm{I}}_{nB}
\tag{2.27}
$$

Similarly, the amplitude form of the rotation after Lorentz transformation matrix

$$
\begin{aligned}
C_B^{ROT^\circ} &= \begin{bmatrix} \cos\phi^\circ - jn_1^\circ \sin\phi^\circ & (-n_3^\circ - jn_2^\circ)\sin\phi^\circ \\ (n_3^\circ - jn_2^\circ)\sin\phi^\circ & \cos\phi^\circ + jn_1^\circ \sin\phi^\circ \end{bmatrix}_B \\
&= \cos\phi^\circ \underline{\sigma}_0 - j(n_1^\circ \underline{\sigma}_1 + n_2^\circ \underline{\sigma}_2 + n_3^\circ \underline{\sigma}_3)_B \sin\phi^\circ \\
&= \exp(-j\underline{n}_B^\circ \phi^\circ) \Leftrightarrow \exp(n_B^\circ \phi^\circ)
\end{aligned}
\tag{2.28}
$$

with the matrix form of the real rotation axis

$$
\underline{n}_B^\circ = (n_1^\circ \underline{\sigma}_1 + n_2^\circ \underline{\sigma}_2 + n_3^\circ \underline{\sigma}_3)_B \equiv (\mathrm{q}\underline{\sigma}_1 + \mathrm{u}\underline{\sigma}_2 + \mathrm{v}\underline{\sigma}_3)_B^{\mathrm{A}^\circ} \Leftrightarrow jn_B^\circ \tag{2.29}
$$

can serve to obtain the quaternionic expression

$$
\begin{aligned}
\mathcal{C}_B^{ROT\circ} &= \cos\phi^\circ + (n_1^\circ e_1 + n_2^\circ e_2 + n_3^\circ e_3)_B \sin\phi^\circ \\
&= \exp(n_B^\circ \phi^\circ)
\end{aligned}
\tag{2.30}
$$

with the quaternion n_B° of the real rotation axis, through an A° point on the Poincare sphere:

$$
n_B^\circ = (n_1^\circ e_1 + n_2^\circ e_2 + n_3^\circ e_3)_B \equiv (\mathrm{q}e_1 + \mathrm{u}e_2 + \mathrm{v}e_3)_B^{\mathrm{A}^\circ} \tag{2.31}
$$

Of course, the angle of rotation, $2\phi^\circ$, is independent of the PP basis, similarly as $\tanh\alpha = (\mathrm{OI})_n$ does, the distance from the inversion point I to the center O of the Poincare sphere of unit radius, the model of the Jones matrix.

2.3 Quaternionic form of the Stokes 4-vector

In any B basis, the complex quaternionic form of the unit Stokes 4-vector of the completely polarized incident wave, represented by a T point on the Poincare sphere, can be found through the density matrix expressed by the PP complex vector u_B^T (see [9]):

$$
\begin{aligned}
\mathcal{P}_B^{\mathrm{T}} &= \tfrac{1}{\sqrt{2}}\{1 + j(\mathrm{q}\,e_1 + \mathrm{u}\,e_2 + \mathrm{v}\,e_3)_B^{\mathrm{T}}\} \\
\Leftrightarrow \sqrt{2}\,u_B^T \tilde{u}_B^T{*} &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1+\mathrm{q} & \mathrm{u} - j\mathrm{v} \\ \mathrm{u} + j\mathrm{v} & 1 - \mathrm{q} \end{bmatrix}_B^{\mathrm{T}} \\
&= \tfrac{1}{\sqrt{2}}\{\underline{\sigma}_0 + (\mathrm{q}\,\underline{\sigma}_1 + \mathrm{u}\,\underline{\sigma}_2 + \mathrm{v}\,\underline{\sigma}_3)_B^{\mathrm{T}}\}
\end{aligned}
\tag{2.32}
$$

where the PP column vector takes the form (same as in the X basis – see Section 1). (Quaternions proposed earlier by Czyz [3] (1967) did not use their complex representation.)

$$u_B^T = \begin{bmatrix} \cos\gamma \exp[-j(\delta+\varepsilon)] \\ \sin\gamma \exp[+j(\delta-\varepsilon)] \end{bmatrix}_B^T \tag{2.33}$$

A digression: The unit quaternion of the Stokes 4-vector for the partially polarized wave will be of similar form

$$\mathcal{P}_B^{\mathrm{T}} = \cos 2\theta + j(\mathrm{q}\, e_1 + \mathrm{u}\, e_2 + \mathrm{v}\, e_3)_B^{\mathrm{T}} \sin 2\theta \tag{2.34}$$

corresponding to the column unit Stokes 4-vector

$$\mathrm{P}_B^{\mathrm{T}} = \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \cos 2\gamma_B \\ \sin 2\theta \sin 2\gamma_B \cos 2\delta_B \\ \sin 2\theta \sin 2\gamma_B \sin 2\delta_B \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \cos 2\theta \\ \sin 2\theta\, \mathrm{q}_B \\ \sin 2\theta\, \mathrm{u}_B \\ \sin 2\theta\, \mathrm{v}_B \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \mathrm{I} \\ \mathrm{Q}_B \\ \mathrm{U}_B \\ \mathrm{V}_B \end{bmatrix}_n^{\mathrm{T}} \tag{2.35}$$

with $\tan 2\theta$ denoting the degree of polarization which, for the completely polarized wave, is equal to $\tan 45^0 = 1$, with $\cos 2\theta = \sin 2\theta = \frac{1}{\sqrt{2}}$. Such a unit 4-vector, or the corresponding quaternion, can be represented by a point on the surface of a hypersphere of unit radius in the 4-dimensional space of Stokes parameters. The crossection of that hypersphere by a hyperplane $\mathrm{I}_n = \frac{1}{\sqrt{2}}$ forms the Poincare sphere of complete polarizations rotated in the 3-dimensional space according to the implied basis B. Such representation of partial polarizations by points in the 4-dimensional space, instead of the usual representation by points inside the Poincare sphere, appears useful when considering reception of partially polarized waves expressed, for example, by dot products (see Section 3.5) of the corresponding quaternions.

2.4 Quaternionic forms of amplitude rotation matrices changing the PP vector (the TP phasor) bases of vectors and matrices, and rotating the PP vectors

Quaternionic form of equation for the change of the TP phasor basis $X \to B$ of the Stokes 4-vector can be found by considering at first the matrix in the X basis changing the corresponding PP vector (see Appendix A):

$$u_X^X \to u_X^B = [u_X^B \quad u_X^{B\mathrm{x}}]\begin{bmatrix}1\\0\end{bmatrix} = C_X^B u_X^X \tag{2.36}$$

Quaternionic forms of such a matrix, rotating the PP vector, and of its inverse corresponding to the appropriate matrices, can be written as follows (see Appendices A and D):

$$\begin{aligned} \mathcal{C}_X^B &= \cos\phi_X^B + (n_1e_1 + n_2e_2 + n_3e_3)_X^{X\to B}\sin\phi_X^B \\ \Leftrightarrow C_X^B &= \cos\phi_X^B \underline{\sigma}_0 - j(n_1\underline{\sigma}_1 + n_2\underline{\sigma}_2 + n_3\underline{\sigma}_3)_X^{X\to B}\sin\phi_X^B \\ \mathcal{C}_B^X &= \widetilde{\mathcal{C}}_X^B{}^* = \cos\phi_X^B - (n_1e_1 + n_2e_2 + n_3e_3)_X^{X\to B}\sin\phi_X^B \\ \Leftrightarrow C_B^X &= \widetilde{C}_X^B{}^* = \cos\phi_X^B\underline{\sigma}_0 + j(n_1\underline{\sigma}_1 + n_2\underline{\sigma}_2 + n_3\underline{\sigma}_3)_X^{X\to B}\sin\phi_X^B \end{aligned} \tag{2.37}$$

The dependences

$$\begin{aligned} \cos\phi_X^B &= \cos\gamma_X^B\cos(\delta+\varepsilon)_X^B \\ n_{1X}^{X\to B}\sin\phi_X^B &= \cos\gamma_X^B\sin(\delta+\varepsilon)_X^B \\ n_{2X}^{X\to B}\sin\phi_X^B &= -\sin\gamma_X^B\sin(\delta-\varepsilon)_X^B \\ n_{3X}^{X\to B}\sin\phi_X^B &= \sin\gamma_X^B\cos(\delta-\varepsilon)_X^B \end{aligned} \tag{2.38}$$

can be found when considering the equalities (see also AppendixA):

$$\begin{aligned} C_X^B &= \left[u_X^B \quad u_X^{B\mathrm{x}}\right] = \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}_X^B = \begin{bmatrix} \cos\gamma_X^B\exp[-(\delta+\varepsilon)_X^B] & -\sin\gamma_X^B\exp[-(\delta-\varepsilon)_X^B] \\ \sin\gamma_X^B\exp(\delta-\varepsilon)_X^B & \cos\gamma_X^B\exp(\delta+\varepsilon)_X^B \end{bmatrix} \\ &= C_X^{ROT}\left(\underline{n}_X^{X\to B}, 2\phi_X^B\right) = \begin{bmatrix} \cos\phi_X^B - jn_{1X}^{X\to B}\sin\phi_X^B & (-n_{3X}^{X\to B} - jn_{2X}^{X\to B})\sin\phi_X^B \\ (n_{3X}^{X\to B} - jn_{2X}^{X\to B})\sin\phi_X^B & \cos\phi_X^B + jn_{1X}^{X\to B}\sin\phi_X^B \end{bmatrix} \end{aligned}$$

(2.39)

For reasons of convenience, the rule has been introduced about symbols of quaternions representing transformation matrices. Symbols of mutually corresponding matrices and quaternions differ *only* by one defining font. For example, in the equivalence of $\widetilde{\mathcal{C}}_X^B{}^* \Leftrightarrow \widetilde{C}_X^B{}^*$, symbols of transposition and complex conjugation being applied to matrices, together with indexes, convey without change to quaternions and only the 'Times New Roman' font C of the matrix has been changed for the 'Lucida Handwritting' $\mathcal{C}$ of the quaternion (both in italics).

Matrix equation for the change of the TP phasor basis

$$u_X^P \to u_B^P = C_B^X u_X^P = \widetilde{C}_X^B{}^* u_X^P \tag{2.40}$$

leads to the following equation for the change of the TP phasor basis of the Stokes polarization quaternion

$$\mathcal{P}_X^{\mathrm{P}} \rightarrow \mathcal{P}_B^{\mathrm{P}} = \tilde{C}_X^B * \mathcal{P}_X^{\mathrm{P}} C_X^B \Leftrightarrow \tilde{C}_X^B * u_X^P (\tilde{u}_X^P) * C_X^B \tag{2.41}$$

Similarly, the equation for the change of the Stokes polarization quaternion itself, corresponding to the matrix equation,

$$u_B^P \rightarrow u_B^K = C_{P,B}^K u_B^P \equiv C_B^K C_P^B u_B^P = C_B^K \tilde{C}_B^P * u_B^P \tag{2.42}$$

is

$$\mathcal{P}_B^{\mathrm{P}} \rightarrow \mathcal{P}_B^{\mathrm{K}} = C_B^K C_P^B \mathcal{P}_B^{\mathrm{P}} C_B^P C_K^B = C_B^K \tilde{C}_B^P * \mathcal{P}_B^{\mathrm{P}} C_B^P \tilde{C}_B^K * \tag{2.43}$$

Quaternionic equation for the change of the TP phasor basis for the Jones matrix, also resembling the appropriate matrix equation, is

$$\mathcal{A}_X^{\circ} \rightarrow \mathcal{A}_B^{\circ} = \tilde{C}_X^B * \mathcal{A}_X^{\circ} C_X^B \tag{2.44}$$

2.5 Quaternionic propagation equation in the form of the product of quaternions of the Lorentz and rotation transformations

The Jones propagation equation

$$\lambda^{\mathrm{T}} u_B^{\mathrm{So}} = A_B^{\circ} u_B^T = e^{j\xi^{\circ}} \frac{\sqrt{\sigma_0}}{2} C_B^{ROT^{\circ}} A_{nB}^{LOR} u_B^T \tag{2.45}$$

can be presented in the quaternionic form through the following mutual relation:

$$\begin{aligned} (\lambda^{\mathrm{T}})^2 \sqrt{2}\, u_B^{\mathrm{So}} \tilde{u}_B^{\mathrm{So}} * &= \frac{\sigma_0}{4} C_B^{ROT^{\circ}} A_{nB}^{LOR} \sqrt{2}\, u_B^T \tilde{u}_B^T * \tilde{A}_{nB}^{LOR} * \tilde{C}_B^{ROT^{\circ}} * \\ \Leftrightarrow (\lambda^{\mathrm{T}})^2 \mathcal{P}_B^{\mathrm{So}} &= \frac{\sigma_0}{4} \mathcal{C}_B^{ROT^{\circ}} \mathcal{A}_{nB}^{LOR} \mathcal{P}_B^{\mathrm{T}} \tilde{\mathcal{A}}_{nB}^{LOR} * \tilde{\mathcal{C}}_B^{ROT^{\circ}} * \end{aligned} \tag{2.46}$$

where

$$\tilde{\mathcal{A}}_{nB}^{LOR} * = \mathcal{A}_{nB}^{LOR} = 1 - j(\mathrm{Q}e_1 + \mathrm{U}e_2 + \mathrm{V}e_3)_{nB}^{\mathrm{I}} \tag{2.47}$$

and

$$\widetilde{\mathcal{C}}_B^{ROT^\circ}* = \cos\phi^\circ - (\mathrm{q}e_1 + \mathrm{u}e_2 + \mathrm{v}e_3)_B^{A^\circ} \sin\phi^\circ = \cos\phi^\circ - (n_1^\circ e_1 + n_2^\circ e_2 + n_3^\circ e_3)_B \sin\phi^\circ$$
$$= \exp(-n_B^\circ \phi^\circ)$$

(2.48)

With $(\lambda^{\mathrm{T}})^2 \equiv \sigma^{\mathrm{T}}$, the result of the quaternionic multiplications can be presented in the form

$$\sigma^{\mathrm{T}} \mathcal{P}_B^{S\circ} = \sigma^{\mathrm{T}} \tfrac{1}{\sqrt{2}} \{1 + j(\mathrm{q}\, e_1 + \mathrm{u}\, e_2 + \mathrm{v}\, e_3)_B^{S^\circ}\}$$
$$= \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & je_1 & je_2 & je_3 \end{bmatrix} \mathrm{K}_B^\circ \mathrm{P}_B^{\mathrm{T}} \qquad (2.49)$$

exactly corresponding to the propagation equation $\sigma^{\mathrm{T}} \mathrm{P}_B^{S\circ} = \mathrm{K}_B^\circ \mathrm{P}_B^{\mathrm{T}}$ with the Mueller matrix $\mathrm{K}_B^\circ = \widetilde{\mathrm{U}} * (A_B^\circ \otimes A_B^\circ *) \mathrm{U}$ (see Appendix E).

2.6 Derivation of the Jones matrix decomposition, in the quaternion form, using the dephased matrix method

The dephased Jones matrix in the quaternion form can be decomposed into product of two quaternions, of the rotation-after-Lorentz transformation and (not normalized, for example) the (pure) Lorentz transformation (Lorentz amplitude transformation without real rotation):

$$\mathcal{A}_B^\circ e^{-j\xi^\circ} = \mathcal{C}_B^{ROT\circ} \mathcal{A}_B^{LOR} \qquad (2.50)$$

what can be checked by expression of all quaternionic factors through elements of the Jones matrix and by accomplishment of quaternion multiplications.

To perform all those manipulations, the quaternionic forms will be useful of the whole Jones matrix itself and its versions: transposed/conjugate, transposed, and conjugate, accordingly, which are:

$$\mathcal{A}_B^\circ = \tfrac{1}{2}\{(A_2^\circ + A_1^\circ)e_0 + j[(A_2^\circ - A_1^\circ)e_1 + (A_3^\circ + A_4^\circ)e_2 + j(A_3^\circ - A_4^\circ)e_3]\}$$
$$= \tfrac{1}{2}\{A_2^\circ + A_1^\circ + j(A_2^\circ - A_1^\circ)e_1 + j(A_3^\circ + A_4^\circ)e_2 - (A_3^\circ - A_4^\circ)e_3\} \qquad (2.51)$$

$$\widetilde{\mathcal{A}}_B^\circ * = \tfrac{1}{2}\{(A_2^\circ * + A_1^\circ *)e_0 + j[(A_2^\circ * - A_1^\circ *)e_1 + (A_3^\circ * + A_4^\circ *)e_2 - j(A_3^\circ * - A_4^\circ *)e_3]\}$$
$$= \tfrac{1}{2}\{A_2^\circ * + A_1^\circ * + j(A_2^\circ * - A_1^\circ *)e_1 + j(A_3^\circ * + A_4^\circ *)e_2 + (A_3^\circ * - A_4^\circ *)e_3\}$$

(2.52)

$$\tilde{A}_B^{\circ} = \tfrac{1}{2}\{A_2^{\circ} + A_1^{\circ} + j(A_2^{\circ} - A_1^{\circ})e_1 + j(A_3^{\circ} + A_4^{\circ})e_2 + (A_3^{\circ} - A_4^{\circ})e_3\} \tag{2.53}$$

$$A_B^{\circ}{*} = \tfrac{1}{2}\{A_2^{\circ}{*} + A_1^{\circ}{*} + j(A_2^{\circ}{*} - A_1^{\circ}{*})e_1 + j(A_3^{\circ}{*} + A_4^{\circ}{*})e_2 - (A_3^{\circ}{*} - A_4^{\circ}{*})e_3\} \tag{2.54}$$

The quaternionic form of the rotation-after-Lorentz transformation in terms of the dephased Jones matrix can be expressed as follows (see Appendix B):

$$C_B^{ROT^{\circ}} = \frac{1}{\sqrt{\sigma_0}}\{A_B^{\circ} e^{-j\xi^{\circ}} - e_3 A_B^{\circ}{*}\, e_3\, e^{j\xi^{\circ}}\} \tag{2.55}$$

That is indeed the rotation-after-Lorentz transformation matrix because with

$$\tilde{C}_B^{ROT^{\circ}}{*} = \frac{1}{\sqrt{\sigma_0}}\{\tilde{A}_B^{\circ}{*}\, e^{j\xi^{\circ}} - e_3 \tilde{A}_B^{\circ} e_3\, e^{-j\xi^{\circ}}\} \tag{2.56}$$

one obtains

$$C_B^{ROT^{\circ}} \tilde{C}_B^{ROT^{\circ}}{*} = \tilde{C}_B^{ROT^{\circ}}{*}\, C_B^{ROT^{\circ}} = 1 \tag{2.57}$$

and because it leads to the quaternionic form of the Lorentz transformation matrix:

$$\begin{aligned}
A_B^{LOR} &= \tilde{C}_B^{ROT\circ}{*}\, A_B^{\circ} e^{-j\xi^{\circ}} \\
&= \frac{1}{\sqrt{\sigma_0}}\{\tilde{A}_B^{\circ}{*} - e_3 \tilde{A}_B^{\circ} e_3\, e^{-2j\xi^{\circ}}\} A_B^{\circ} \\
&= \frac{1}{\sqrt{\sigma_0}}\{\tilde{A}_B^{\circ}{*}\, A_B^{\circ} + \mathrm{a}_0\} \\
&= \frac{1}{\sqrt{\sigma_0}}\{\mathrm{a}_1 + \mathrm{a}_0 + j(\mathrm{b}_1 e_1 + \mathrm{b}_3 e_2 - j\mathrm{b}_5 e_3)_B\} \\
&= \frac{\sqrt{\sigma_0}}{2}\left\{\frac{2(\mathrm{a}_1 + \mathrm{a}_0)}{\sigma_0} + j\left(\frac{2\mathrm{b}_1}{\sigma_0}e_1 + \frac{2\mathrm{b}_3}{\sigma_0}e_2 - j\frac{2\mathrm{b}_5}{\sigma_0}e_3\right)_B\right\} \\
&= \frac{\sqrt{\sigma_0}}{2}\left\{1 - j\left(\mathrm{Q}_n^1 e_1 + \mathrm{U}_n^1 e_2 + \mathrm{V}_n^1 e_3\right)_B\right\}
\end{aligned} \tag{2.58}$$

and to its normalized version,

$$\begin{aligned}
A_{nB}^{LOR} &= \frac{2}{\sigma_0}\{\mathrm{a}_1 + \mathrm{a}_0 + j(\mathrm{b}_1 e_1 + \mathrm{b}_3 e_2 - j\mathrm{b}_5 e_3)_B \\
&= 1 - j\left(\mathrm{Q}_n^1 e_1 + \mathrm{U}_n^1 e_2 + \mathrm{V}_n^1 e_3\right)_B
\end{aligned} \tag{2.59}$$

2.7 Poincare sphere geometrical model of the Jones propagation quaternion (matrix)

Simple geometrical constructions present the Poincare sphere model of the Jones propagation matrix (or quaternion). Two special functions of the inversion point in such a model should be observed:

- determination of the emerging intensity,
- explanation of the polarization and phase transformation.

When taking the diameter of the sphere equal to $\sqrt{\sigma_0}$, the outgoing power for any polarization point T of the incoming wave of unit power equals the square of the distance of that point from the inversion point I:

$$\sigma^{\mathrm{T}} = (\mathrm{IT})^2 = \frac{\sigma_0}{4}(\mathrm{IT})_n^2 \tag{2.60}$$

Such a result can be obtained, for example, when considering transformation of any PP unit vector by the normalized amplitude Lorentz matrix in an M basis corresponding to the polarization point M and with the M phasor oriented along equator in the plane through the points I and T in direction to the T point:

$$\begin{aligned}\begin{bmatrix}1-\mathrm{Q} & 0\\ 0 & 1+\mathrm{Q}\end{bmatrix}_M^{\mathrm{I}}\begin{bmatrix}a\\ b\end{bmatrix}_M^T &\equiv \begin{bmatrix}1+(\mathrm{OI})_n & 0\\ 0 & 1-(\mathrm{OI})_n\end{bmatrix}\begin{bmatrix}\cos\gamma\\ \sin\gamma\end{bmatrix}_M^T\\ &= \begin{bmatrix}[1+(\mathrm{OI})_n]\cos\gamma_M^T\\ [1-(\mathrm{OI})_n]\sin\gamma_M^T\end{bmatrix} = \lambda_n^T\begin{bmatrix}\cos\gamma\\ \sin\gamma\end{bmatrix}_M^{T^*}\end{aligned} \tag{2.61}$$

That leads to

$$\lambda_n^{\mathrm{T}} = \sqrt{1+(\mathrm{OI})_n^2 + 2(\mathrm{OI})_n\cos 2\gamma_M^T} = \sqrt{\sigma_n^{\mathrm{T}}} \tag{2.62}$$

or

$$\sigma_n^{\mathrm{T}} = 1+(\mathrm{OI})_n^2 + 2(\mathrm{OI})_n\cos 2\gamma_{\mathrm{M}}^{\mathrm{T}} = (\mathrm{IT})_n^2 \tag{2.63}$$

what agrees with the Carnot Theorem for the triangle OIT inside the Poincare sphere of unit radius (for $(\mathrm{IM})_n = 1+(\mathrm{OI})_n$ and $2\gamma_{\mathrm{M}}^{\mathrm{T}}$ being an angle between the radii $(\mathrm{OT})_n$ and $(\mathrm{OM})_n$).

Additionally, the result of the above normalized Lorentz transformation determines the following relation between polarization ratios of the input and output waves

:

$$\tan\gamma_{\mathrm{M}}^{\mathrm{T}^*} = \frac{1-(\mathrm{OI})_n}{1+(\mathrm{OI})_n}\tan\gamma_{\mathrm{M}}^{\mathrm{T}} \tag{2.64}$$

That result follows also similarity of triangles NTI and MT'I, as well as NIT' and MIT, with the TT' chord through the I point. It resembles the Kennaugh's proof of similar geometrical problem for the inversion matrix (compare his M.Sc. work, p. 21, eqn. (75)). In this case, however, it is dealing with the Lorentz matrix. It allows to obtain the $T'' = T'\times$ point, for the T point chosen, by inversion $T \to T'$ (through the point I back to the sphere) followed by orthogonality $T' \to T''$, geometrically denoting inversion through the center point O of the sphere. It is worth noticing that the Lorentz transformation does not change the phase.

Similarity of triangles NIT' and MIT leads also to

$$(\mathrm{IT'})_n = \frac{1-(\mathrm{OI})_n^2}{\lambda_n^{\mathrm{T}}} \tag{2.65}$$

and for $\phi'' \equiv \gamma_M^{\mathrm{T}} - \gamma_M^{\mathrm{T''}}$, with the T'T'' representing diameter of the sphere, presents

$$\cos\phi'' = \frac{(\mathrm{IT'})_n + (\mathrm{IT})_n}{2} = \frac{1-(\mathrm{OI})_n^2 + (\lambda_n^{\mathrm{T}})^2}{2\lambda_n^{\mathrm{T}}} \tag{2.66}$$

Altogether, two explanations for the amplitude Lorentz transformation exist: the 'hyperbolic' (or equivalent real) $T \to T''$ rotation, or the successive inversion and orthogonality transformation.

According to its second function, the inversion point I enables one to find the polarization point So of the outgoing wave. To obtain that result one can, for example,

- to rotate the sphere in the (previously determined) M basis about the axis $v_M = -1$ by the angle $2\phi'' \equiv 2\phi_T^{T''} = 2\gamma_M^T - 2\gamma_M^{T''}$, dependent on the T phasor, to obtain the T'' phasor (after the Lorentz transformation), and then
- to rotate the sphere about the n_M° axis by the $2\phi^{\circ}$ angle.

It is interesting to add that also the last axis and angle of rotation can be obtained having known the inversion point coordinates in a special basis, characteristic for the Jones matrix. That agrees with the fact that the dephased Jones matrix depends on 7 real parameters only. Four of them are the Poincare sphere model diameter plus three coordinates of the inversion point, and the three remaining are, for example, three Euler angles by which the sphere should be rotated to its 'characteristic' orientation (compare similar considerations referred to the Sinclair matrix in Czyz's ONR-Report-3, Section 9.7 and Appendices G - J).

To arrive at the So phasor, the additional rotation of such rotated phasor is required, by the $-2\xi^{\circ}$ angle about the OSo axis.

The first of the rotations mentioned above, the real rotation by the $2\phi''$ angle, corresponds to the imaginary rotation determined by the Lorentz matrix. It depends on the T point chosen and replaces the amplitude Lorentz matrix according to the equalities:

$$\begin{aligned} A_{nM}^{LOR} &= \lambda^T C_{T,M}^{T''}; \\ C_{T,M}^{T''} &= C_{T,M}^{T''}(n_{3M} = -1,\ 2\phi'') \equiv C_M^{T''} C_T^M; \\ C_{T,M}^{T''} u_M^T &= u_M^{T''} \end{aligned} \tag{2.67}$$

The last transformation can be rewritten as follows

$$\begin{bmatrix} \cos\phi'' & \sin\phi'' \\ -\sin\phi'' & \cos\phi'' \end{bmatrix} \begin{bmatrix} \cos\gamma \\ \sin\gamma \end{bmatrix}_M^T = \begin{bmatrix} \cos\gamma_M^T \cos\phi'' + \sin\gamma_M^T \sin\phi'' \\ \sin\gamma_M^T \cos\phi'' - \cos\gamma_M^T \sin\phi'' \end{bmatrix} = \begin{bmatrix} \cos(\gamma_M^T - \phi'') \\ \sin(\gamma_M^T - \phi'') \end{bmatrix} = \begin{bmatrix} \cos\gamma_M^{T''} \\ \sin\gamma_M^{T''} \end{bmatrix} \tag{2.68}$$

with the previously obtained angle ϕ''.

3. RADAR POLARIMETRY. THE SCATTERING TRANSFORMATION

A digression about the E (or 'amplitude') representation of the electric field vector (compare: Pellat-Finet [8], Optik, 1991, pp.27, and 70).

The electric vector's plane wave *propagating in the* $+z$ *direction* of the local right-handed xyz spatial frame can be written in different forms, depending on the choice of the orthogonal basis of collinear phasors, X or B, in which its angular parameters are being expressed:

$$\begin{aligned} {}^{+}\boldsymbol{E}^P(t,z) &= E_0^P \begin{bmatrix} \boldsymbol{1}_x & \boldsymbol{1}_y \end{bmatrix} \mathrm{Re}\{u_X^P \exp[j(\omega t - kz)]\} \\ &= E_0^P \begin{bmatrix} \boldsymbol{1}_x & \boldsymbol{1}_y \end{bmatrix} \mathrm{Re}\{(\cos\gamma_X^P u_X^X + \sin\gamma_X^P u_X^{Xx} \exp(j2\delta_X^P)) \exp[j(\omega t - (kz + v_X^P))]\} \end{aligned} \tag{3.1}$$

or in terms of the B basis column PP vectors

$$\begin{aligned} {}^{+}\boldsymbol{E}^P(t,z) &= E_0^P \begin{bmatrix} \boldsymbol{1}_x & \boldsymbol{1}_y \end{bmatrix} \mathrm{Re}\{C_X^B u_B^P \exp[j(\omega t - kz)]\} \\ &= E_0^P \begin{bmatrix} \boldsymbol{1}_x & \boldsymbol{1}_y \end{bmatrix} \mathrm{Re}\{(\cos\gamma_B^P u_X^B + \sin\gamma_B^P u_X^{Bx} \exp(j2\delta_B^P)) \exp[j(\omega t - (kz + v_B^P))]\} \end{aligned} \tag{3.2}$$

with

$$u_X^X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad u_X^{Xx} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad v_X^P = \delta_X^P + \varepsilon_X^P, \quad v_B^P = \delta_B^P + \varepsilon_B^P \tag{3.3}$$

$$
\begin{aligned}
u_X^B &= (\cos\gamma_X^B \begin{bmatrix}1\\0\end{bmatrix} + \sin\gamma_X^B \exp(j2\delta_X^B)\begin{bmatrix}0\\1\end{bmatrix})\exp(-j\nu_X^B) \\
&= \begin{bmatrix}\cos\gamma_X^B \\ \sin\gamma_X^B \exp(j2\delta_X^B)\end{bmatrix}\exp(-j\nu_X^B)
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
u_X^{Bx} &= (-\sin\gamma_X^B \exp(-j2\delta_X^B)\begin{bmatrix}1\\0\end{bmatrix} + \cos\gamma_X^B \begin{bmatrix}0\\1\end{bmatrix})\exp(j\nu_X^B) \\
&= \begin{bmatrix}-\sin\gamma_X^B \exp(-j2\delta_X^B) \\ \cos\gamma_X^B\end{bmatrix}\exp(j\nu_X^B)
\end{aligned}
\tag{3.5}
$$

This can be written because the change-of-basis rule (see Appendix A): $u_X^P = \begin{bmatrix}u_X^X & u_X^{Xx}\end{bmatrix} u_X^P$

$$= \begin{bmatrix}u_X^B & u_X^{Bx}\end{bmatrix} u_B^P = C_X^B u_B^P \tag{3.6}$$

The same wave, when changing the propagation direction for the opposite one (*and when applying the 'time reversal' concept*) can be successively written as follows:

$$
\begin{aligned}
{}^{-}\boldsymbol{E}^P(t,z) &= E_0^P\begin{bmatrix}\boldsymbol{1}_x & \boldsymbol{1}_y\end{bmatrix}\mathrm{Re}\{u_X^P \exp[j(-\omega t - kz)]\} = E_0^P\begin{bmatrix}\boldsymbol{1}_x & \boldsymbol{1}_y\end{bmatrix}\mathrm{Re}\{(u_X^P)^* \exp[j(\omega t + kz)]\} \\
&= E_0^P\begin{bmatrix}\boldsymbol{1}_x & \boldsymbol{1}_y\end{bmatrix}\mathrm{Re}\{(\cos\gamma_X^P u_X^X + \sin\gamma_X^P u_X^{Xx}\exp(-j2\delta_X^P))\exp[j(\omega t + (kz + \nu_X^P))]\}
\end{aligned}
\tag{3.7}
$$

or in terms of the B basis conjugate column PP vectors:

$$
\begin{aligned}
{}^{-}\boldsymbol{E}^P(t,z) &= E_0^P\begin{bmatrix}\boldsymbol{1}_x & \boldsymbol{1}_y\end{bmatrix}\mathrm{Re}\{C_X^B u_B^P \exp[j(-\omega t - kz)]\} = E_0^P\begin{bmatrix}\boldsymbol{1}_x & \boldsymbol{1}_y\end{bmatrix}\mathrm{Re}\{(C_X^B u_B^P)^* \exp[j(\omega t + kz)]\} \\
&= E_0^P\begin{bmatrix}\boldsymbol{1}_x & \boldsymbol{1}_y\end{bmatrix}\mathrm{Re}\{(\cos\gamma_B^P (u_X^B)^* + \sin\gamma_B^P (u_X^{Bx})^* \exp(-j2\delta_B^P))\exp[j(\omega t + (kz + \nu_B^P))]\}
\end{aligned}
\tag{3.8}
$$

Both waves have the same polarization, represented by the P point on the Poincare sphere, and their equations present two polarization helices, coinciding at the moment of time $t = 0$ but moving in opposite directions. Position of such a helix in space at the time $t = 0$, expressed versus the z axis (the propagation axis), can be called its *spatial phase*, for example ν_X^P or ν_B^P, eventually ε_X^P or ε_B^P, depending on the definition applied and on the basis assumed. However, senses of polarization ellipses traced by both waves penetrating the $z = 0$ plane are different! Therefore it is not advisable, *in radar applications*, to define the polarization by the polarization ellipse. Instead, the *polarization helix* together with its spatial phase, both determined by the same TP phasor P on the polarization sphere, is the best solution to that problem. In time, such a phasor rotates in opposite directions depending on the direction of propagation, $\pm z$.

3.1 Spatial frame reversal-by-rotation and orthogonality transformations

The Sinclair ('scattering') matrix can be obtained from the Jones ('propagation') matrix through the reversal-by-rotation of the output local spatial frame. That results in the output *complex amplitude* (CA) unit vector represented by the *conjugate* PP vector, u_B^S*, of the scattered wave:

$$A_B u_B^T = \lambda^T u_B^S *; \qquad A_B = C_B^\circ A_B^\circ \tag{3.9}$$

In the case of the Sinclair matrix, the emerging wave propagates in the $-z$ direction of the local spatial frame, 'reversed-by-rotation' versus that applied for the Jones matrix.

The transformation equation for the spatial frame reversal by the 180^0 rotation about the y axis is of similar form as that with the A_B matrix (see also Appendix A):

$$C_B^\circ u_B^{S\circ} = u_B^S *; \quad C_B^\circ = \begin{bmatrix} -\mathrm{w} & \mathrm{u} \\ \mathrm{u} & \mathrm{w}* \end{bmatrix}_X^B, \quad \det C_B^\circ = -1, \tag{3.10}$$
$$\mathrm{w}_X^B = (a^2 - b^2)_X^B, \quad \mathrm{u}_X^B = (ab* + ba*)_X^B$$

For the C_X° matrix, its parameters are: w=1, and u=0.

Other special properties of such a 'Sinclair matrix', C_B°, which can be called the 'free space scattering matrix', are: its symmetry, and $\lambda_n^T = 1$, because the inversion point is in the *center* of its normalized Poincare sphere model. That is the reason for applying the "*C*" symbol for that matrix.

Both matrices transform identically under the change of the PP basis:

$$A_B = \widetilde{C}_X^B A_X C_X^B, \qquad C_B^\circ = \widetilde{C}_X^B C_X^\circ C_X^B \tag{3.11}$$

what can be explained by inspection of the transmission equation:

$$V_r = \widetilde{u}_B^R A_B u_B^T = (\widetilde{u}_B^R \widetilde{C}_B^X)(\widetilde{C}_X^B A_B C_B^X)(C_X^B u_B^T) = \widetilde{u}_X^R A_X u_X^T \tag{3.12}$$

Only the orthogonality transformation, also of the same form,

$$C^{\times} u_X^P * = u_X^{P\times}; \quad C^{\times} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{3.13}$$

does not depend on the PP basis because

$$\tilde{C}_X^B C^{\times} C_X^B = C^{\times} \tag{3.14}$$

The following quaternionic expressions can be found: for the reversal-by-rotation matrix, for its conjugate form, and for the orthogonality matrix,

$$\mathcal{C}_B^{\circ} = j(-\operatorname{Im} w - \operatorname{Re} w\, e_1 + \mathrm{u} e_2)_X^B \tag{3.15}$$

$$\mathcal{C}_B^{\circ} * = j(\operatorname{Im} w - \operatorname{Re} w\, e_1 + \mathrm{u} e_2)_X^B \tag{3.16}$$

$$\mathcal{C}^{\times} = e_3 \tag{3.17}$$

In scattering, the transformation equations in the quaternionic form should take into account conjugate values of some PP vectors. Therefore, to arrive at the Stokes polarization quaternion in the reversed-by-rotation spatial frame, one should write

$$\sqrt{2}\, u_B^S \tilde{u}_B^S * = C_B^{\circ} * \sqrt{2}\, u_B^{S\circ} * \tilde{u}_B^{S\circ} C_B^{\circ} \Leftrightarrow \mathcal{P}_B^{S+} = \mathcal{C}_B^{\circ} * \mathcal{P}_B^{S\circ -} \mathcal{C}_B^{\circ} \tag{3.18}$$

where:

$$\mathcal{P}_B^{S+} = \tfrac{1}{\sqrt{2}}\{1 + j(\mathrm{q}\, e_1 + \mathrm{u}\, e_2 + \mathrm{v}\, e_3)_B^S\} \tag{3.19}$$

and

$$\mathcal{P}_B^{S\circ -} = \tfrac{1}{\sqrt{2}}\{1 + j(\mathrm{q}\, e_1 + \mathrm{u}\, e_2 - \mathrm{v}\, e_3)_B^{S\circ}\} \tag{3.20}$$

Those quaternions correspond to the complex amplitude (CA) vectors of waves propagating in the $+z$ and $-z$ directions of the local spatial frame, accordingly. Only the 'CA quaternion' of a wave propagating in the $+z$ direction equals the Stokes polarization quaternion,

$$\mathcal{P}_B^{S+} \equiv \mathcal{P}_B^{S} \tag{3.21}$$

Therefore, in the case of the 'scattered' emerging waves which propagate in the $-z$ direction of the local spatial frame (contrary to the 'propagated' emerging waves represented by the CA *and* PP $u_B^{S\circ}$ vectors) their Stokes polarization quaternions correspond to the CA vectors, u_B^S,

for the hypothetically reversed direction of propagation of those waves. After all quaternionic multiplications one obtains the end result:

$$\mathcal{P}_B^S = \begin{bmatrix}1 & je_1 & je_2 & je_3\end{bmatrix} \mathrm{D}_B^{\circ} \mathrm{P}_B^{\mathrm{S}\circ} = \tfrac{1}{\sqrt{2}}\{1 + j(\mathrm{q}\, e_1 + \mathrm{u}\, e_2 + \mathrm{v}\, e_3)_B^{\mathrm{S}}\} \tag{3.22}$$

where

$$\mathrm{P}_B^{\mathrm{S}\circ} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ \mathrm{q}\\ \mathrm{u}\\ \mathrm{v}\end{bmatrix}_B^{\mathrm{S}\circ} \quad \text{and} \quad \mathrm{D}_B^{\circ} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1-2\mathrm{u}^2 & -2\mathrm{u}\,\mathrm{Re}\,\mathrm{w} & -2\mathrm{u}\,\mathrm{Im}\,\mathrm{w}\\ 0 & -2\mathrm{u}\,\mathrm{Re}\,\mathrm{w} & \mathrm{u}^2 - \mathrm{Re}\,\mathrm{w}^2 & -\mathrm{Im}\,\mathrm{w}^2\\ 0 & -2\mathrm{u}\,\mathrm{Im}\,\mathrm{w} & -\mathrm{Im}\,\mathrm{w}^2 & \mathrm{u}^2 + \mathrm{Re}\,\mathrm{w}^2\end{bmatrix}_X^B \tag{3.23}$$

Similarly, the orthogonality transformation is

$$\begin{aligned}\mathcal{P}_B^{\mathrm{P}\times} &= \mathcal{C}^{\times}\mathcal{P}_B^{\mathrm{P}-}\tilde{\mathcal{C}}^{\times} = e_3 \tfrac{1}{\sqrt{2}}\{1 + j(\mathrm{q}\, e_1 + \mathrm{u}\, e_2 - \mathrm{v}\, e_3)_B^{\mathrm{P}}\}(-e_3)\\ &= \tfrac{1}{\sqrt{2}}\{1 - j(\mathrm{q}\, e_1 + \mathrm{u}\, e_2 + \mathrm{v}\, e_3)_B^{\mathrm{P}}\}\end{aligned} \tag{3.24}$$

3.2 Derivation of the Sinclair matrix decomposition in the quaternion form using the dephased matrix method

The quaternionic form of the rotation-after-inversion transformation matrix in terms of the dephased Sinclair matrix (compare Appendix B) is:

$$\mathcal{C}_B^{ROT} * = \frac{1}{\sqrt{\sigma_0}}\{\mathcal{A}_B\, e^{-j\xi}\, e_3 + e_3 \mathcal{A}_B * e^{j\xi}\} \tag{3.25}$$

with its inverse

$$\tilde{\mathcal{C}}_B^{ROT} = \frac{1}{\sqrt{\sigma_0}}\{-e_3 \tilde{\mathcal{A}}_B * e^{j\xi} - \tilde{\mathcal{A}}_B\, e_3\, e^{-j\xi}\} \tag{3.26}$$

This is indeed the rotation-after-inversion transformation because

$$\mathcal{C}_B^{ROT} * \tilde{\mathcal{C}}_B^{ROT} = \tilde{\mathcal{C}}_B^{ROT} \mathcal{C}_B^{ROT} * = 1 \tag{3.27}$$

and because it leads to the quaternionic form of the inversion transformation which corresponds to the Lorentz transformation followed by the inversion through the center of the Poincare sphere, the inverse versus the orthogonality transformation:

$$
\begin{aligned}
\mathcal{A}_B^{INV} &= \widetilde{C}_B^{ROT} \mathcal{A}_B e^{-j\xi} \\
&= \frac{1}{\sqrt{\sigma_0}}\{-e_3 \widetilde{\mathcal{A}}_B{}^* - \widetilde{\mathcal{A}}_B\, e_3\, e^{-2j\xi}\}\mathcal{A}_B \\
&= -e_3 \frac{1}{\sqrt{\sigma_0}}\{\widetilde{\mathcal{A}}_B{}^* - e_3\, \widetilde{\mathcal{A}}_B\, e_3\, e^{-2j\xi}\}\mathcal{A}_B
\end{aligned}
\tag{3.28}
$$

When searching the derivation of the Lorentz matrix observe please that the form of the inversion matrix, or its quaternion, must not depend on the choice of the Sinclair or Jones matrices taken together with their phase terms. So:

$$
\begin{aligned}
\mathcal{A}_B^{INV} &= -e_3 \frac{1}{\sqrt{\sigma_0}}\{\widetilde{\mathcal{A}}_B^{\circ}{}^* - e_3\, \widetilde{\mathcal{A}}_B^{\circ} e_3\, e^{-2j\xi^{\circ}}\}\mathcal{A}_B^{\circ} \\
&= -e_3 \mathcal{A}_B^{LOR}
\end{aligned}
\tag{3.29}
$$

This is because (see also Section 3.4):

$$
\{\widetilde{\mathcal{A}}_B^{\circ}{}^* - e_3 \widetilde{\mathcal{A}}_B^{\circ} e_3\, e^{-2j\xi^{\circ}}\}\mathcal{A}_B^{\circ} = \mathrm{a}_1 + \mathrm{a}_0 + j(\mathrm{b}_1 e_1 + \mathrm{b}_3 e_2 - j\mathrm{b}_5 e_3)_B = \{\widetilde{\mathcal{A}}_B{}^* - e_3\, \widetilde{\mathcal{A}}_B\, e_3\, e^{-2j\xi}\}\mathcal{A}_B
\tag{3.30}
$$

The normalized version of the inversion matrix quaternion becomes

$$
\begin{aligned}
\mathcal{A}_{nB}^{INV} &= -e_3 \frac{2}{\sigma_0}\{\mathrm{a}_1 + \mathrm{a}_0 + j(\mathrm{b}_1 e_1 + \mathrm{b}_3 e_2 - j\mathrm{b}_5 e_3)_B \\
&= -e_3\{1 - j\left(\mathrm{Q}_n^{\mathrm{I}} e_1 + \mathrm{U}_n^{\mathrm{I}} e_2 + \mathrm{V}_n^{\mathrm{I}} e_3\right)_B\} \\
&= -e_3 \mathcal{A}_{nB}^{LOR}
\end{aligned}
\tag{3.31}
$$

3.3 The dependence of the inversion and rotation-after-inversion matrices on elements of the Sinclair matrix

In order to find mutual dependences between elements of the Sinclair matrix and the axis and angle of rotation after the inversion transformation, it is instructive to decompose that matrix as follows (see Czyz's ONR-Report-5, July 31, 2002, p.26, Eqs. (3.60-61)):

$$A_B = \begin{bmatrix} A_2 & A_3 \\ A_4 & A_1 \end{bmatrix}_B = e^{j\xi} \frac{\sqrt{\sigma_0}}{2} C_B^{ROT} * A_{nB}^{INV} \tag{3.32}$$

with

$$\begin{aligned} C_B^{ROT} * &= \frac{1}{\sqrt{\sigma_0}} \left\{ \left(A_B e^{-j\xi} \right) C^\times + C^\times \left(A_B * e^{j\xi} \right) \right\} \\ &= \frac{1}{\sqrt{\sigma_0}} \begin{bmatrix} A_3 e^{-j\xi} - A_4 * e^{j\xi} & -A_2 e^{-j\xi} - A_1 * e^{j\xi} \\ A_1 e^{-j\xi} + A_2 * e^{j\xi} & -A_4 e^{-j\xi} + A_3 * e^{j\xi} \end{bmatrix}_B \\ &= \begin{bmatrix} \cos\phi + jn_1 \sin\phi & (-n_3 + jn_2)\sin\phi \\ (n_3 + jn_2)\sin\phi & \cos\phi - jn_1 \sin\phi \end{bmatrix}_B \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} A_{nB}^{INV} &= e^{-j\xi} \frac{2}{\sqrt{\sigma_0}} \widetilde{C}_B^{ROT} A_B = \frac{2}{\sigma_0} \left\{ \widetilde{C}^\times \widetilde{A}_B * + \widetilde{A}_B \widetilde{C}^\times e^{-2j\xi} \right\} A_B \\ &= \frac{2}{\sigma_0} \begin{bmatrix} A_2 A_3 * + A_4 A_1 * & M_3 + M_1 + |\det A| \\ -M_2 - M_4 - |\det A| & -A_3 A_2 * - A_1 A_4 * \end{bmatrix}_B \\ &= \frac{2}{\sigma_0} \begin{bmatrix} \mathrm{b}_3 + j\mathrm{b}_5 & \frac{\sigma_0}{2} + \mathrm{b}_1 \\ -\frac{\sigma_0}{2} - \mathrm{b}_1 & -\mathrm{b}_3 + j\mathrm{b}_5 \end{bmatrix}_B \\ &= \begin{bmatrix} -\mathrm{U} - j\mathrm{V} & 1 + \mathrm{Q} \\ -1 + \mathrm{Q} & \mathrm{U} - j\mathrm{V} \end{bmatrix}_{nB}^{\mathrm{I}} \end{aligned} \tag{3.34}$$

By comparing the two last forms of the amplitude matrix of rotation-after-inversion, one obtains:

$$\begin{bmatrix} \cos\phi \\ n_{1B} \sin\phi \\ n_{2B} \sin\phi \\ n_{3B} \sin\phi \end{bmatrix} = \frac{1}{\sqrt{\sigma_0}} \begin{bmatrix} \mathrm{Re}\{(A_3 - A_4)_B \exp(-j\xi)\} \\ \mathrm{Im}\{(A_3 + A_4)_B \exp(-j\xi)\} \\ \mathrm{Im}\{(A_1 - A_2)_B \exp(-j\xi)\} \\ \mathrm{Re}\{(A_1 + A_2)_B \exp(-j\xi)\} \end{bmatrix} \tag{3.35}$$

Of course, the angle of rotation after inversion, 2ϕ, does not depend on the B basis. This is because the difference of elements of the Sinclair matrix second diagonal is independent of that PP basis.

To summarize mutual dependences between matrices of decompositions of the Jones and Sinclair matrices, it is instructive to write (see Appendix B, formulae (B.3) and (B.5)):

$$C_B^{ROT}* = -jC_B^{\circ}C^{\times}C_B^{ROT^{\circ}}* = -j\widetilde{C}^{\times}C_B^{\circ}*C_B^{ROT^{\circ}}* = -jC_B^{\circ}C_B^{ROT^{\circ}}C^{\times}$$
$$A_B^{INV} = \widetilde{C}^{\times}A_B^{LOR} = -C^{\times}A_B^{LOR} \tag{3.36}$$

That is an immediate result of the equalities:

$$A_B = C_B^{\circ}A_B^{\circ} = C_B^{\circ}(e^{j\xi^{\circ}}C_B^{ROT^{\circ}}A_B^{LOR}) = je^{j\xi^{\circ}}(-jC_B^{\circ}C_B^{ROT^{\circ}}C^{\times})(\widetilde{C}^{\times}A_B^{LOR}) = e^{j\xi}C_B^{ROT}*A_B^{INV} \tag{3.37}$$

with

$$e^{j\xi} = je^{j\xi^{\circ}} \leftrightarrow \xi = \xi^{\circ} + \tfrac{\pi}{2} \tag{3.38}$$

3.4 Similarities and differences of the Poincare sphere models of the Sinclair and Jones matrices

For

$$A_B^{\circ} = \begin{bmatrix} A_2^{\circ} & A_3^{\circ} \\ A_4^{\circ} & A_1^{\circ} \end{bmatrix}_B = C_B^{\circ}*A_B \tag{3.39}$$

$$A_B = \begin{bmatrix} A_2 & A_3 \\ A_4 & A_1 \end{bmatrix}_B = C_B^{\circ}A_B^{\circ} \tag{3.40}$$

and

$$\mathrm{M}_{kB}^{\circ} = A_{kB}^{\circ}A_{kB}^{\circ}*, \quad \mathrm{M}_{kB} = A_{kB}A_{kB}*; \quad k = 1,2,3,4 \tag{3.41}$$

it can be checked immediately that (see also Appendix E):

$$\begin{gathered}
\mathrm{M}_{2B}^{\circ} + \mathrm{M}_{4B}^{\circ} = \mathrm{M}_{2B} + \mathrm{M}_{4B} = \mathrm{a}_1 + \mathrm{b}_{1B} \\
\mathrm{M}_{3B}^{\circ} + \mathrm{M}_{1B}^{\circ} = \mathrm{M}_{3B} + \mathrm{M}_{1B} = \mathrm{a}_1 - \mathrm{b}_{1B} \\
A_{2B}^{\circ}A_{3B}^{\circ}* + A_{4B}^{\circ}A_{1B}^{\circ}* = A_{2B}A_{3B}* + A_{4B}A_{1B}* = \mathrm{b}_{3B} + j\mathrm{b}_{5B} \\
|A_{2B}^{\circ}A_{1B}^{\circ} - A_{3B}^{\circ}A_{4B}^{\circ}| = |A_{2B}A_{1B} - A_{3B}A_{4B}| = |\det A_B^{\circ}| = |\det A_B| = \mathrm{a}_0
\end{gathered} \tag{3.42}$$

what results in

$$\begin{gathered}
\tfrac{1}{2}(\mathrm{M}_{2B}^{\circ} + \mathrm{M}_{3B}^{\circ} + \mathrm{M}_{4B}^{\circ} + \mathrm{M}_{1B}^{\circ}) = \tfrac{1}{2}(\mathrm{M}_{2B} + \mathrm{M}_{3B} + \mathrm{M}_{4B} + \mathrm{M}_{1B}) = \mathrm{a}_1 \\
\tfrac{1}{2}(\mathrm{M}_{2B}^{\circ} - \mathrm{M}_{3B}^{\circ} + \mathrm{M}_{4B}^{\circ} - \mathrm{M}_{1B}^{\circ}) = \tfrac{1}{2}(\mathrm{M}_{2B} - \mathrm{M}_{3B} + \mathrm{M}_{4B} - \mathrm{M}_{1B}) = \mathrm{b}_{1B}
\end{gathered} \tag{3.43}$$

and means that the Jones and Sinclair matrices have the Poincare sphere models of the same diameter,

$$\sqrt{\sigma_0} = \sqrt{2(\mathrm{a}_1 + \mathrm{a}_0)} \tag{3.44}$$

and the same coordinates of the I and M points in the same B basis (compare (2.27)):

$$\begin{bmatrix} \mathrm{Q} \\ \mathrm{U} \\ \mathrm{V} \end{bmatrix}_{nB}^{\mathrm{I}} = \frac{-2}{\sigma_0}\begin{bmatrix} \mathrm{b}_1 \\ \mathrm{b}_3 \\ \mathrm{b}_5 \end{bmatrix}_B = -\tanh\alpha \begin{bmatrix} \mathrm{q} \\ \mathrm{u} \\ \mathrm{v} \end{bmatrix}_B^{\mathrm{M}} \tag{3.45}$$

with the same independent of the B basis parameter

$$\tanh\alpha = 2\sqrt{\mathrm{b}_{1B}^2 + \mathrm{b}_{3B}^2 + \mathrm{b}_{5B}^2}\,/\,\sigma_0 \equiv 2\mathrm{b}_0\,/\,\sigma_0 = (\mathrm{OI})_n \tag{3.46}$$

Results obtained are physically obvious because emerging power cannot depend on the spatial frame reversal for the outgoing wave. Also they indicate the α angle to be a common, intrinsic parameter of the Jones and Sinclair matrices, uniquely determining the inversion I point coordinates of which depend on the PP basis B, arbitrarily chosen. That inversion point determines, in a different way, the inversion and the Lorentz matrices/quaternions being used to build the Sinclair and Jones matrices/quaternions, respectively.

Contrary to that, the rotation-after inversion and the rotation-after-Lorentz transformation matrices/quaternions are different. Their axes and angles of rotation depend on the reversal-by-rotation transformation of the spatial frame for the outgoing wave. General dependences, for any PP basis B, joining axes and angles $n_B, 2\phi$ with $n_B^\circ, 2\phi^\circ$ are as follows:

$$\begin{bmatrix} \cos\phi^\circ \\ n_1^\circ \sin\phi^\circ \\ n_2^\circ \sin\phi^\circ \\ n_3^\circ \sin\phi^\circ \end{bmatrix}_B = \begin{bmatrix} -(n_{1B}\mathrm{u}_X^B + n_{2B}\operatorname{Re}\mathrm{w}_X^B + n_{3B}\operatorname{Im}\mathrm{w}_X^B)\sin\phi \\ \mathrm{u}_X^B\cos\phi + (n_{3B}\operatorname{Re}\mathrm{w}_X^B - n_{2B}\operatorname{Im}\mathrm{w}_X^B)\sin\phi \\ \operatorname{Re}\mathrm{w}_X^B\cos\phi + (n_{1B}\operatorname{Im}\mathrm{w}_X^B - n_{3B}\mathrm{u}_X^B)\sin\phi \\ \operatorname{Im}\mathrm{w}_X^B\cos\phi + (n_{2B}\mathrm{u}_X^B - n_{1B}\operatorname{Re}\mathrm{w}_X^B)\sin\phi \end{bmatrix} \tag{3.47}$$

They directly correspond to the quaternionic form of the spatial frame reversal-by-rotation matrix

$$\begin{aligned} C_B^{ROT\circ} &= jC^{\times} C_B^{\circ} C_B^{ROT} \\ &= je_3[j(-\operatorname{Im}\mathrm{w} - \operatorname{Re}\mathrm{w}\, e_1 + \mathrm{u}e_2)_X^B][\cos\phi + (n_1e_1 + n_2e_2 + n_3e_3)_B \sin\phi] \\ &= (\mathrm{u}e_1 + \operatorname{Re}\mathrm{w}\, e_2 + \operatorname{Im}\mathrm{w}\, e_3)_X^B[\cos\phi + (n_1e_1 + n_2e_2 + n_3e_3)_B \sin\phi] \end{aligned} \tag{3.48}$$

$$\begin{aligned} &= -(n_{1B}\mathrm{u}_X^B + n_{2B}\operatorname{Re}\mathrm{w}_X^B + n_{3B}\operatorname{Im}\mathrm{w}_X^B)\sin\phi \\ &+ e_1[\mathrm{u}_X^B\cos\phi + (n_{3B}\operatorname{Re}\mathrm{w}_X^B - n_{2B}\operatorname{Im}\mathrm{w}_X^B)\sin\phi] \\ &+ e_2[\operatorname{Re}\mathrm{w}_X^B\cos\phi + (n_{1B}\operatorname{Im}\mathrm{w}_X^B - n_{3B}\mathrm{u}_X^B)\sin\phi] \\ &+ e_3[\operatorname{Im}\mathrm{w}_X^B\cos\phi + (n_{2B}\mathrm{u}_X^B - n_{1B}\operatorname{Re}\mathrm{w}_X^B)\sin\phi] \end{aligned} \tag{3.49}$$

$$= \cos\phi^\circ + (n_{1B}^\circ e_1 + n_{2B}^\circ e_2 + n_{3B}^\circ e_{31})\sin\phi^\circ$$

In the special case $B=X$ (in the linear PP basis X, where w=1 and u=0) one obtains:

$$\begin{bmatrix} \cos\phi^\circ \\ n_1^\circ \sin\phi^\circ \\ n_2^\circ \sin\phi^\circ \\ n_3^\circ \sin\phi^\circ \end{bmatrix}_B = \begin{bmatrix} -n_{2X}\sin\phi \\ n_{3X}\sin\phi \\ \cos\phi \\ -n_{1X}\sin\phi \end{bmatrix} = \frac{1}{\sqrt{\sigma_0}}\begin{bmatrix} -\operatorname{Im}[(A_1 - A_2)_X e^{-j\xi}] \\ \operatorname{Re}[(A_1 + A_2)_X e^{-j\xi}] \\ \operatorname{Re}[(A_3 - A_4)_X e^{-j\xi}] \\ -\operatorname{Im}[(A_3 + A_4)_X e^{-j\xi}] \end{bmatrix} = \frac{1}{\sqrt{\sigma_0}}\begin{bmatrix} \operatorname{Re}[(A_2^\circ + A_1^\circ)_X e^{-j\xi^\circ}] \\ -\operatorname{Im}[(A_2^\circ - A_1^\circ)_X e^{-j\xi^\circ}] \\ -\operatorname{Im}[(A_4^\circ + A_3^\circ)_X e^{-j\xi^\circ}] \\ \operatorname{Re}[(A_4^\circ - A_3^\circ)_X e^{-j\xi^\circ}] \end{bmatrix} \tag{3.50}$$

3.5 Reception of the scattered power

The power received by an antenna from a scattered wave can be expressed by the 'dot product' of two Stokes polarization quaternions, one of the antenna and another one of the scattered wave. Such expression follows its corresponding matrix form:

$$\begin{aligned} P_r &= |\tilde{u}_B^R u_B^{S*}|^2 \\ &= \tilde{\mathrm{P}}_B^R \mathrm{P}_\mathrm{B}^S = \tfrac{1}{2}(1 + \mathrm{q}_B^R\mathrm{q}_B^S + \mathrm{u}_B^R\mathrm{u}_B^S + \mathrm{v}_B^R\mathrm{v}_B^S) \\ &= \mathcal{P}_B^R \bullet \mathcal{P}_\mathrm{B}^S \end{aligned} \tag{3.51}$$

In case of the partially polarized scattered wave the received power expressed by the product of the Stokes polarization vectors/quaternions is (when applying forms (1.4) and (2.35)):

$$P = \tilde{\mathrm{P}}_B^R \mathrm{P}_\mathrm{B}^S = \mathcal{P}_B^R \bullet \mathcal{P}_\mathrm{B}^S = \tfrac{1}{\sqrt{2}}\{\cos 2\theta^S + (\mathrm{q}_B^R\mathrm{q}_B^S + \mathrm{u}_B^R\mathrm{u}_B^S + \mathrm{v}_B^R\mathrm{v}_B^S)\sin 2\theta^S\} \tag{3.51a}$$

4. DIRECT TRANSMISSION BETWEEN ANTENNAS

When considering the direct transmission between antennas it seems natural to express the PP vectors of those antennas, or their corresponding Stokes polarization quaternions, in their 'own' local spatial frames *xyz*, with the *z* axes directed outward, 'from antennas' looking at each other, and with the *y* axes remaining parallel (the last condition is suitable for applying the C_X° matrix, or the $\mathcal{C}_X^{\circ}$ quaternion, in the here assumed form). Following such assumptions, the received power can be presented in the form:

$$\begin{aligned} P_r &= |\tilde{u}_B^R C_B^{\circ} u_B^T|^2 = \tilde{\mathrm{P}}_B^R \mathrm{D}_{\mathrm{B}}^{\circ} \mathrm{P}_{\mathrm{B}}^T = \tilde{\mathrm{P}}_B^R \mathrm{P}_{\mathrm{B}}^{T\circ} = \tilde{\mathrm{P}}_B^{R\circ} \mathrm{P}_{\mathrm{B}}^T \\ &= \mathcal{P}_B^R \bullet \mathcal{P}_{\mathrm{B}}^{T\circ} = \mathcal{P}_B^{R\circ} \bullet \mathcal{P}_{\mathrm{B}}^T \end{aligned} \tag{4.1}$$

Here, the *polarization* quaternion of one antenna, for example corresponding to the u_B^R as its unit complex height or its polarization and phase vector, is

$$\mathcal{P}_B^R \Leftrightarrow \sqrt{2} u_B^R \tilde{u}_B^R * = \frac{1}{\sqrt{2}} \begin{bmatrix} 1+\mathrm{q} & \mathrm{u}-j\mathrm{v} \\ \mathrm{u}+j\mathrm{v} & 1-\mathrm{q} \end{bmatrix}_B^R \tag{4.2}$$

The Stokes quaternion corresponding to the *conjugate* complex height of another antenna,

$$\mathcal{P}_B^{T-} \Leftrightarrow \sqrt{2} u_B^T * \tilde{u}_B^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1+\mathrm{q} & \mathrm{u}+j\mathrm{v} \\ \mathrm{u}-j\mathrm{v} & 1-\mathrm{q} \end{bmatrix}_B^T \rightarrow \mathcal{P}_B^{T-} = \tfrac{1}{\sqrt{2}}\{1 + j(\mathrm{q}\, e_1 + \mathrm{u}\, e_2 - \mathrm{v}\, e_3)_B^T\} \tag{4.3}$$

(which is not a polarization quaternion), determines the *polarization* quaternion of that antenna when expressed in the spatial coordinate system reversed, by rotation, versus its 'own' local spatial frame:

$$\begin{aligned} \mathcal{P}_B^{T\circ} &\Leftrightarrow \sqrt{2} u_B^{T\circ} \tilde{u}_B^{T\circ} * = \frac{1}{\sqrt{2}} \begin{bmatrix} 1+\mathrm{q} & \mathrm{u}-j\mathrm{v} \\ \mathrm{u}+j\mathrm{v} & 1-\mathrm{q} \end{bmatrix}_B^{T\circ} \rightarrow \mathcal{P}_B^{T\circ} = \tfrac{1}{\sqrt{2}}\{1 + j(\mathrm{q}\, e_1 + \mathrm{u}\, e_2 + \mathrm{v}\, e_3)_B^{T\circ}\} \\ &= \mathcal{C}_B^{\circ} * \mathcal{P}_B^{T-} \mathcal{C}_B^{\circ} \\ &= j(\mathrm{Im}\,\mathrm{w} - \mathrm{Re}\,\mathrm{w}\, e_1 + \mathrm{u} e_2)_X^B \tfrac{1}{\sqrt{2}}\{1 + j(\mathrm{q}\, e_1 + \mathrm{u}\, e_2 - \mathrm{v}\, e_3)_B^T\} j(-\mathrm{Im}\,\mathrm{w} - \mathrm{Re}\,\mathrm{w}\, e_1 + \mathrm{u} e_2)_X^B \\ &= \tfrac{1}{\sqrt{2}}\{1 + j e_1[(1-2\mathrm{u}^2)_X^B \mathrm{q}_B^T - (2\mathrm{u}\,\mathrm{Re}\,\mathrm{w})_X^B \mathrm{u}_B^T - (2\mathrm{u}\,\mathrm{Im}\,\mathrm{w})_X^B \mathrm{v}_B^T] \\ &\quad + j e_2[-(2\mathrm{u}\,\mathrm{Re}\,\mathrm{w})_X^B \mathrm{q}_B^T + (\mathrm{u}^2 - \mathrm{Re}\,\mathrm{w}^2)_X^B \mathrm{u}_B^T - (\mathrm{Im}\,\mathrm{w}^2{}_X^B \mathrm{v}_B^T] \\ &\quad + j e_3[-(2\mathrm{u}\,\mathrm{Im}\,\mathrm{w})_X^B \mathrm{q}_B^T - (\mathrm{Im}\,\mathrm{w}^2)_X^B \mathrm{u}_B^T + (\mathrm{u}^2 + \mathrm{Re}\,\mathrm{w}^2)_X^B \mathrm{v}_B^T] \end{aligned} \tag{4.4}$$

Observe, please, that physically the same polarization, emitted by the same antenna in the same direction, can be presented by two different polarization 4-vectors (quaternions) when expressed in mutually reversed (by rotation) spatial coordinate systems. It confirms the rule that description of polarization of an antenna or wave is always referred to its spatial frame.

For the linear X basis of collinear phasors, or for the left circular (LC, RC) basis of parallel phasors, for which w = 1 and u = 0, one obtains the known result,

$$
\begin{aligned}
P_r &= (\mathcal{P}^R \bullet \mathcal{P}^{T\circ})_{X \text{ or } (LC,RC)} = (\mathcal{P}^{R\circ} \bullet \mathcal{P}^T)_{X \text{ or } (LC,RC)} \\
&= \tfrac{1}{2}(1 + \mathrm{q}^R\mathrm{q}^T - \mathrm{u}^R\mathrm{u}^T + \mathrm{v}^R\mathrm{v}^T)_{X \text{ or } (LC,RC)}
\end{aligned}
\tag{4.5}
$$

(About the parallel phasor basis see: the Czyz's paper: "A comparison of different kinds of orthogonal polarimetric bases" in MIKON 2002 Conf. Proc., pp.164-167, or his ONR-Report-3, pp.53-54.)

It should be stressed that the 'dot product' of polarization quaternions in the above presented formulae is different from that proposed in [9] or [10] because it is identical to the Euklidean scalar product of Stokes four-vectors or to the real part (in quaternionic sense) of the complex quaternions' product. So, in applications to transmission between antennas or to reception of waves, complex quaternions corresponding to the Stokes 4-vectors *are not* elements of the Minkowski space with Lorentz metric. They undergo the Cartesian metric, also for hypercone of complete polarizations resembling the light cone in the theory of special relativity. Nevertheless, they undergo Lorentz transformations discussed in [10] as vectors in relativistic kinematics.

5. GEOMETRICAL MODELS OF THE PROPAGATION AND SCATTERING AMPLITUDE MATRICES AND THEIR COMPARISON

Models of matrices are tangential polarization phasor (TPP) spheres of given diameter, inversion point, axis of rotation, angle of rotation about that axis, first phasor, B say, of the orthogonal collinear basis, and angle of the 'general' phase. Common to both propagation (Jones, A_B°) and scattering (Sinclair, A_B) matrices, corresponding to the same obstacle being met by the wave on its path, is diameter of the sphere, $\sqrt{\sigma_0}$, the inversion point, I, and (usually) the basis assumed in which elements of matrices and three coordinates of the I point are determined. Different are: the way of using the inversion point, and the axes and angles of

rotation. General phase differs by 90^0 for the basis assumed. First transformation of the incident wave's TP phasor is with the use of the I point, then comes rotation.

The simplest is the inversion transformation, being applied when considering scattering, determined by the inversion matrix, A_B^{INV} . It moves the polarization point T of the incident wave by projection through the inversion point I back to the sphere, to the point T'=T INV. Direction of the transformed phasor, tangent at that point, can be determined by the following procedure. The two points, T and T', should be joined by the oriented shorter great circle arc in such a way that its origin is placed at the inverted point, T', and its end at the incident polarization point, T. At first, the incident phasor should be rotated to become collinear and identically oriented with the T'T arc, then it should be shifted 'back' to the point T' and rotated again in the same direction by the same angle. After such procedure it will be oriented at the same angle but of opposite sign versus the T'T arc in comparison with its previous direction. Analytically, such result follows the conjugate value of the inverted PP vector equal to CA of the transformed wave. At the end the phasor should be multiplied by the real number λ^T equal to the IT distance inside the sphere of diameter $\sqrt{\sigma_0}$.

More complex is the Lorentz transformation being applied when considering the propagation transformation and determined by the Lorentz matrix, A_B^{LOR} . At first it operates like the inversion matrix but afterward it realizes the orthogonality which is the reversed transformation versus inversion through the center point O of the sphere. If the I and O points coincide then the incident phasor remains unchanged after the Lorentz transformation. Multiplication of the resultant phasor by the term λ^T is also necessary.

Lorentz transformation can be considered also as rotation, apart from multiplication by λ^T, but then axis and angle of such rotation depend on the incident polarization point. Axis of that rotation is perpendicular to the great circle plane through the I and T points. That axis is also perpendicular to the 'imaginary rotation' axis oriented along the sphere diameter through the I point. The imaginary rotation axis is independent of the T point location. There are two rotation matrices corresponding to models of the Jones and Sinclair matrices: $C_B^{ROT^\circ}$ and C_B^{ROT} *, accordingly.

However, there is a possibility to use conjugate versions of those matrices, $C_B^{ROT^\circ}$ * and C_B^{ROT} , after regular inversion and Lorentz transformations in other Sinclair and Jones

matrices (see Appendix C). Necessity of using those other scattering and propagation matrices becomes obvious when considering, for example, the regular Jones matrix for which there is a need to find its special polarization points. One can examine the other Sinclair matrix, $A_B^{xo} = \widetilde{C}^x A_B^{o}$, for the same obstacle in order to find special polarization points for that matrix using well known methods applicable to scattering matrices. Then the corresponding special polarization points for the original regular Jones matrix can be immediately found by applying to that 'scattered' phasor, $-\lambda^T u_B^{Sox}$, the orthogonality transformation according to the rule:

$$\begin{aligned} A_B^{o} u_B^T &= C^x A_B^{xo} u_B^T = C^x \widetilde{C}^x A_B^{o} u_B^T \\ &= \lambda^T C^x \widetilde{C}^x u_B^{So} = -\lambda^T C^x (u_B^{Sox})^* = -\lambda^T u_B^{Soxx} = \lambda^T u_B^{So} \end{aligned} \tag{5.1}$$

What one has to realize is that special polarization points of the A_B^{xo} Sinclair matrix change their meaning after orthogonalization procedure. Points representing incident polarizations corresponding to maximum emerging power exchange their location with those producing minimum output power, and vice versa. Next, co-pol-nulls exchange their meaning with cross-pol-nulls what means that in the case of propagation the cross-pol-nulls (eigenpolarizations) always do exist, as co-pol-nulls for scattering, while the polarizations which become orthogonal may sometimes disappear.

Interesting is the case of the scattering matrix model in the so called characteristic basis *K*. Complex elements of its second diagonal are of opposite sign (their sum disappears) and elements of the first diagonal are of the same phase (as elements of symmetrical matrix taking diagonal form in that *K* basis). It means that for opposite direction of transmission the model rotates by 180^0 about its OQ_K axis (together with all special polarization points). The axis of rotation after inversion is always situated in the $\mathrm{U}_K\mathrm{V}_K$ plane, and the chord joining co-pol-null points is parallel to the V_K axis. There is an allowed region inside the Poincaré sphere model in which the inversion I point can be situated. Using coordinates of the I point the whole Sinclair matrix in the *K* basis (except of the general phase) can be restored; in a part of that allowed region two solutions exist leading to two different Sinclair matrices corresponding to the same I point. Depending on the number of solution and on location of the I point inside the allowed region different mutual locations of special polarization points result. Most interesting are mutual locations of those points for the inversion I point situated on the boundary surfaces of the allowed regions for the two solutions. Special geometrical

constructions have been developed enabling one to find special polarization points for assumed coordinates of the I point located on boundary surfaces of its allowed regions or in coordinate planes of the Poincaré sphere corresponding to the characteristic K basis (see [3]).

The nonsymmetrical Sinclair matrix model is much more complicated than the model of symmetrical scattering matrix considered by Kennaugh. Kennaugh's model has dealt with one solution only for the scattering matrix, and the allowed region for the I point in his case was only the negative part of the OQ_K axis including the center of the sphere (for the so called 'symmetrical' targets) and the point on its surface (for the 'dipole type' targets).

6. CONCLUDING REMARKS

Presented work extends the Kennaugh's inversion point concept to the case of Jones and nonsymmetrical Sinclair matrices. Polarization and phase vectors of waves interpreted as tangential polarization phasors have been used to construct geometrical models of those matrices in the form of the Poincaré spheres of the same diameter and equipped with common inversion point but different axes and angles of rotation. Propagation (optical) and scattering (radar) transformations using Jones and Sinclair matrices, accordingly, have been presented in both matrix and quaternionic forms exhibiting full agreement of the two approaches. The choice of the approach can be left to researchers who will apply one more suitable to solve particular problems.

APPENDIX A. COLUMN PP VECTOR TRANSFORMATIONS

Passive TPP sphere rotation transformation - change of the PP vector (or TP phasor) basis

Denoting: $\boldsymbol{u}^X = [\boldsymbol{1}_x \quad \boldsymbol{1}_y] u_X^X$, $\boldsymbol{u}^{Xx} = [\boldsymbol{1}_x \quad \boldsymbol{1}_y] u_X^{Xx}$, and $u_X^X \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $u_X^{Xx} \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, one obtains $\boldsymbol{1}_x = \boldsymbol{u}^X$ and $\boldsymbol{1}_y = \boldsymbol{u}^{Xx}$. Therefore:

$$\begin{aligned} \boldsymbol{u}^P &= [\boldsymbol{1}_x \quad \boldsymbol{1}_y] u_X^P = [\boldsymbol{u}^X \quad \boldsymbol{u}^{Xx}] u_X^P \\ &= [\boldsymbol{u}^B \quad \boldsymbol{u}^{Bx}] u_B^P = [[\boldsymbol{u}^X \quad \boldsymbol{u}^{Xx}] u_X^B \quad [\boldsymbol{u}^X \quad \boldsymbol{u}^{Xx}] u_X^{Bx}] u_B^P \\ &= [\boldsymbol{u}^X \quad \boldsymbol{u}^{Xx}][u_X^B \quad u_X^{Bx}] u_B^P \\ &= [\boldsymbol{u}^X \quad \boldsymbol{u}^{Xx}] C_X^B u_B^P \;\Rightarrow\; u_B^P \to u_X^P = C_X^B u_B^P \end{aligned} \tag{A.1}$$

where

$$C_X^B = [u_X^B \quad u_X^{Bx}] = \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}_X^B \tag{A.2}$$

Active rotation transformation – change of the PP vector (or TP phasor) by rotation of the TPP sphere

$$u_B^P \to u_B^K = C_{P,B}^K u_B^P \quad \text{if} \quad C_{P,B}^K = C_B^K C_P^B \tag{A.3}$$

Indeed:

$$C_B^K C_P^B u_B^P = C_B^K u_P^P = [u_B^K \quad u_B^{Kx}] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = u_B^K \tag{A.4}$$

Let

$$C_{P,A}^K = \begin{bmatrix} \exp(-j\phi_P^K) & 0 \\ 0 & \exp(+j\phi_P^K) \end{bmatrix} \tag{A.5}$$

then:

$$\begin{aligned} C_P^K = C_{P,P}^K = C_P^A C_{P,A}^K C_A^P &= \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}_P^A \begin{bmatrix} \exp(-j\phi_P^K) & 0 \\ 0 & \exp(+j\phi_P^K) \end{bmatrix} \begin{bmatrix} a^* & b^* \\ -b & a \end{bmatrix}_P^A \\ &= \begin{bmatrix} aa^*\exp(-j\phi_P^K) + bb^*\exp(+j\phi_P^K) & ab^*\exp(-j\phi_P^K) - ab^*\exp(+j\phi_P^K) \\ a^*b\exp(-j\phi_P^K) - a^*b\exp(+j\phi_P^K) & -bb^*\exp(-j\phi_P^K) + aa^*\exp(+j\phi_P^K) \end{bmatrix}_P^A \\ &= \begin{bmatrix} \cos\phi_P^K - j\mathrm{q}_P^A \sin\phi_P^K & (-\mathrm{v}_P^A - j\mathrm{u}_P^A)\sin\phi_P^K \\ (\mathrm{v}_P^A - j\mathrm{u}_P^A)\sin\phi_P^K & \cos\phi_P^K + j\mathrm{q}_P^A \sin\phi_P^K \end{bmatrix} \end{aligned} \tag{A.6}$$

Also

$$C_P^K = C_{P,K}^K = C_K^A C_{P,A}^K C_A^K = \begin{bmatrix} \cos\phi_P^K - j\mathrm{q}_K^A \sin\phi_P^K & (-\mathrm{v}_K^A - j\mathrm{u}_K^A)\sin\phi_P^K \\ (\mathrm{v}_K^A - j\mathrm{u}_K^A)\sin\phi_P^K & \cos\phi_P^K + j\mathrm{q}_K^A \sin\phi_P^K \end{bmatrix} \tag{A.7}$$

Therefore, components of the unit axis vector for the real rotation are:

$$\begin{bmatrix} \mathrm{q} \\ \mathrm{u} \\ \mathrm{v} \end{bmatrix}_P^A = \begin{bmatrix} \mathrm{q} \\ \mathrm{u} \\ \mathrm{v} \end{bmatrix}_K^A = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}_{P \text{ or } K}^{A \text{ or } P \to K} \tag{A.8}$$

Spatial reversal transformation (passive or active, by the spatial 180^0 rotation about the y axis)

Such transformation, passive or active, is of the same form and can be presented as

$$u_B^P \to u_B^{P\circ} = (C_B^\circ u_B^P)^* \quad \text{and/or} \quad u_B^{P\circ} \to u_B^P = (C_B^\circ u_B^{P\circ})^* \tag{A.9}$$

When applied twice, it brings the vector to its original form,

$$u_B^P = C_B^\circ {}^* C_B^\circ u_B^P \quad \text{and/or} \quad u_B^{P\circ} = C_B^\circ {}^* C_B^\circ u_B^{P\circ} \tag{A.10}$$

where

$$C_B^\circ = \begin{bmatrix} -\mathrm{w} & \mathrm{u} \\ \mathrm{u} & \mathrm{w}^* \end{bmatrix}_X^B = \tilde{C}_X^B C_X^\circ C_X^B; \quad C_X^\circ = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{A.11}$$

with

$$\begin{aligned} \mathrm{u} = \mathrm{u}_X^B &= (ab^* + ba^*)_X^B \\ &= \sin 2\gamma_X^B \cos 2\delta_X^B \end{aligned} \tag{A.12}$$

and

$$\begin{aligned} \mathrm{w} = \mathrm{w}_X^B &= (a^2 - b^2)_X^B \\ &= \cos 2\gamma_X^B \cos 2\delta_X^B \cos 2\varepsilon_X^B - \sin 2\delta_X^B \sin 2\varepsilon_X^B \\ &\quad - j\left(\cos 2\gamma_X^B \cos 2\delta_X^B \sin 2\varepsilon_X^B + \sin 2\delta_X^B \cos 2\varepsilon_X^B\right) \end{aligned} \tag{A.13}$$

Jones and Sinclair matrices, as well as Lorentz and inversion matrices, both represent active transformations.

APPENDIX B. USEFUL FORMULAE FOR RELATIONS BETWEEN SOME SIMPLE AMPLITUDE MATRICES

For the following matrices,

$$C^{\times} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad C^{\circ} \equiv C_B^{\circ} = \begin{bmatrix} -\mathrm{w} & \mathrm{u} \\ \mathrm{u} & \mathrm{w}^* \end{bmatrix}_X^B; \ \det C^{\circ} = -1, \quad C^{ROT} \equiv C_B^{ROT} = \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}_B; \ \det C^{ROT} = 1 \tag{B.1}$$

satisfying relations

$$\begin{gathered} \widetilde{C}^{\times} C^{\times} = C^{\circ} * C^{\circ} = \widetilde{C}^{ROT} * C^{ROT} = diag\{1,1\} \\ \widetilde{C}^{\times} = -C^{\times}, \quad \widetilde{C}^{\circ} = C^{\circ} \end{gathered} \tag{B.2}$$

easy to check products are worth noticing:

$$\begin{aligned} C^{\times} C^{ROT} &= C^{ROT} * C^{\times} \\ C^{\times} C^{ROT} * &= C^{ROT} C^{\times} \\ \widetilde{C}^{\times} C^{ROT} &= C^{ROT} * \widetilde{C}^{\times} \\ \widetilde{C}^{\times} C^{ROT} * &= C^{ROT} \widetilde{C}^{\times} \end{aligned} \quad \text{and} \quad \begin{aligned} C^{\times} C^{\circ} &= C^{\circ} * \widetilde{C}^{\times} \\ C^{\times} C^{\circ} * &= C^{\circ} \widetilde{C}^{\times} \\ \widetilde{C}^{\times} C^{\circ} &= C^{\circ} * C^{\times} \\ \widetilde{C}^{\times} C^{\circ} * &= C^{\circ} C^{\times} \end{aligned} \tag{B.3}$$

$$\begin{aligned} C^{\times} C^{ROT} \widetilde{C}^{\times} = \widetilde{C}^{\times} C^{ROT} C^{\times} &= C^{ROT} * \\ C^{\times} C^{ROT} * \widetilde{C}^{\times} = \widetilde{C}^{\times} C^{ROT} * C^{\times} &= C^{ROT} \end{aligned} \qquad \begin{aligned} C^{\times} C^{\circ} C^{\times} = \widetilde{C}^{\times} C^{\circ} \widetilde{C}^{\times} &= C^{\circ} * \\ C^{\times} C^{\circ} * C^{\times} = \widetilde{C}^{\times} C^{\circ} * \widetilde{C}^{\times} &= C^{\circ} \end{aligned} \tag{B.4}$$

Similarly, for:

$$\begin{aligned} A_B^{LOR} &= C^{\times} A_B^{INV}; \qquad A_{nB}^{LOR} = \frac{2}{\sqrt{\sigma_0}} A_B^{LOR} = \begin{bmatrix} 1-\mathrm{Q} & -\mathrm{U}+j\mathrm{V} \\ -\mathrm{U}-j\mathrm{V} & 1+\mathrm{Q} \end{bmatrix}_{nB}^{1} \\ A_B^{INV} &= \widetilde{C}^{\times} A_B^{LOR}; \qquad A_{nB}^{INV} = \frac{2}{\sqrt{\sigma_0}} A_B^{INV} = \begin{bmatrix} -\mathrm{U}-j\mathrm{V} & 1+\mathrm{Q} \\ -1+\mathrm{Q} & \mathrm{U}-j\mathrm{V} \end{bmatrix}_{nB}^{1} \end{aligned} \tag{B.5}$$

in any B basis one obtains:

$$A_{nB}^{LOR} + \widetilde{C}^{\times} A_{nB}^{LOR} * C^{\times} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_{nB}^{INV} C^{\times} + C^{\times} A_{nB}^{INV} * = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{B.6}$$

Therefore, from definitions of the Jones and Sinclair matrices:

$$A_B^{\circ} = e^{j\xi^{\circ}} \frac{\sqrt{\sigma_0}}{2} C_B^{ROT^{\circ}} A_{nB}^{LOR} \quad \text{and} \quad A_B = e^{j\xi} \frac{\sqrt{\sigma_0}}{2} C_B^{ROT} * A_{nB}^{LOR} \tag{B.7}$$

through identities:

$$\frac{2}{\sqrt{\sigma_0}}\left(A_B^{\circ} e^{-j\xi^{\circ}}\right) = C_B^{ROT^{\circ}} A_{nB}^{LOR},$$

$$\widetilde{C}^{\times}\frac{2}{\sqrt{\sigma_0}}\left(A_B^{\circ} * e^{j\xi^{\circ}}\right)C^{\times} = \widetilde{C}^{\times} C_B^{ROT^{\circ}} * A_{nB}^{LOR} * C^{\times} = C_B^{ROT^{\circ}} \widetilde{C}^{\times} A_{nB}^{LOR} * C^{\times} \tag{B.8}$$

and

$$\frac{2}{\sqrt{\sigma_0}}\left(A_B e^{-j\xi}\right)C^{\times} = C_B^{ROT} * A_{nB}^{INV} C^{\times}, \quad \frac{2}{\sqrt{\sigma_0}} C^{\times}\left(A_B * e^{j\xi}\right) = C^{\times} C_B^{ROT} A_{nB}^{INV} * = C_B^{ROT} * C^{\times} A_{nB}^{INV} * \tag{B.9}$$

one immediately obtains:

$$\frac{1}{\sqrt{\sigma_0}}\left\{\left(A_B^{\circ} e^{-j\xi^{\circ}}\right) + C^{\times}\left(A_B^{\circ} * e^{j\xi^{\circ}}\right)\widetilde{C}^{\times}\right\} = C_B^{ROT^{\circ}} \text{ and } \frac{1}{\sqrt{\sigma_0}}\left\{\left(A_B e^{-j\xi}\right)C^{\times} + C^{\times}\left(A_B * e^{j\xi}\right)\right\} = C_B^{ROT} * \tag{B.10}$$

APPENDIX C. DIFFERENT JONES AND SINCLAIR MATRICES AND THEIR MUTUAL RELATIONS

When applying relations of Appendix B, the following representations of different Jones and Sinclair matrices, useful for applications, can be obtained;

Original and modified Jones matrices:

$$\begin{aligned} A_B^{\circ} &= e^{j\xi^{\circ}} \frac{\sqrt{\sigma_0}}{2} C_B^{ROT^{\circ}} A_{nB}^{LOR} = C^{\times} A_B^{\times\circ} = C_B^{\circ} * A_B = C_B^{\circ} * \widetilde{C}^{\times} A_B^{\times}; \quad A_B^{\circ} u_B^T = \lambda^T u_B^{S\circ} \\ A_B^{\times} &= e^{j\xi} \frac{\sqrt{\sigma_0}}{2} C_B^{ROT} A_{nB}^{LOR} = C^{\times} A_B = C_B^{\circ} * A_B^{\times\circ} = C_B^{\circ} * \widetilde{C}^{\times} A_B^{\circ}; \quad A_B^{\times} u_B^T = \lambda^T u_B^{S\times} \end{aligned} \tag{C.1}$$

Original and modified Sinclair matrices:

$$\begin{aligned} A_B &= e^{j\xi} \frac{\sqrt{\sigma_0}}{2} C_B^{ROT} * A_{nB}^{INV} = \widetilde{C}^{\times} A_B^{\times} = C_B^{\circ} A_B^{\circ} = C_B^{\circ} C^{\times} A_B^{\times\circ}; \qquad A_B u_B^T = \lambda^T (u_B^S) * \\ A_B^{\times\circ} &= e^{j\xi^{\circ}} \frac{\sqrt{\sigma_0}}{2} C_B^{ROT^{\circ}} * A_{nB}^{INV} = \widetilde{C}^{\times} A_B^{\circ} = C_B^{\circ} A_B^{\times} = C_B^{\circ} C^{\times} A_B = -A_B^{\circ\times}; \\ A_B^{\times\circ} u_B^T &= \lambda^T (u_B^{S\times\circ})*; \qquad u_B^{S\times\circ} = -u_B^{S\circ\times} \end{aligned} \tag{C.2}$$

with mutual relations: concerning phase factors,

$$e^{j\xi^{\circ}} = -je^{j\xi}; \quad e^{j\xi} = je^{j\xi^{\circ}} \tag{C.3}$$

rotation matrices (active transformations),

$$\begin{aligned} C_B^{ROT^\circ} &= -jC_B^\circ * C_B^{ROT} * C^\times = jC^\times C_B^\circ C_B^{ROT} \\ C_B^{ROT} * &= -jC_B^\circ C_B^{ROT^\circ} C^\times = jC^\times C_B^\circ * C_B^{ROT^\circ} * \end{aligned} \tag{C.4}$$

and (active) transformations employing the inversion I point,

$$\mathcal{A}_{nB}^{LOR} = C^\times \mathcal{A}_{nB}^{INV}; \qquad \mathcal{A}_{nB}^{INV} = \widetilde{C}^\times \mathcal{A}_{nB}^{LOR} \tag{C.5}$$

APPENDIX D. SOME USEFUL QUATERNIONIC FORMULAE

Rotation quaternions:

$$\begin{aligned} \mathcal{C}_X^B &= \cos\phi_X^B + (n_1e_1 + n_2e_2 + n_3e_3)_X^{X\to B} \sin\phi_X^B \\ \widetilde{\mathcal{C}}_X^B * &= \cos\phi_X^B - (n_1e_1 + n_2e_2 + n_3e_3)_X^{X\to B} \sin\phi_X^B \\ \widetilde{\mathcal{C}}_X^B &= \cos\phi_X^B + (n_1e_1 + n_2e_2 - n_3e_3)_X^{X\to B} \sin\phi_X^B \\ \mathcal{C}_X^B * &= \cos\phi_X^B - (n_1e_1 + n_2e_2 - n_3e_3)_X^{X\to B} \sin\phi_X^B \end{aligned} \tag{D.1}$$

Lorentz and inversion transformation quaternions:

$$\begin{aligned} \mathcal{A}_{nB}^{LOR} &= \widetilde{\mathcal{A}}_{nB}^{LOR} * = 1 - j(\mathrm{Q}e_1 + \mathrm{U}e_2 + \mathrm{V}e_3)_{nB}^{\mathrm{I}} \\ \mathcal{A}_{nB}^{LOR} * &= \widetilde{\mathcal{A}}_{nB}^{LOR} = 1 - j(\mathrm{Q}e_1 + \mathrm{U}e_2 - \mathrm{V}e_3)_{nB}^{\mathrm{I}} \end{aligned} \tag{D.2}$$

$$\begin{aligned} \mathcal{A}_{nB}^{INV} &= -e_3\mathcal{A}_{nB}^{LOR} = -e_3\{1 - j\left(\mathrm{Q}_n^{\mathrm{I}}e_1 + \mathrm{U}_n^{\mathrm{I}}e_2 + \mathrm{V}_n^{\mathrm{I}}e_3\right)_B\} \\ \widetilde{\mathcal{A}}_{nB}^{INV} &= \mathcal{A}_{nB}^{LOR}e_3 = \{1 - j\left(\mathrm{Q}_n^{\mathrm{I}}e_1 + \mathrm{U}_n^{\mathrm{I}}e_2 + \mathrm{V}_n^{\mathrm{I}}e_3\right)_B\}e_3 \\ \mathcal{A}_{nB}^{INV} * &= -e_3\mathcal{A}_{nB}^{LOR} * = -e_3\{1 - j\left(\mathrm{Q}_n^{\mathrm{I}}e_1 + \mathrm{U}_n^{\mathrm{I}}e_2 - \mathrm{V}_n^{\mathrm{I}}e_3\right)_B\} \\ \widetilde{\mathcal{A}}_{nB}^{INV} * &= \mathcal{A}_{nB}^{LOR} * e_3 = \{1 - j\left(\mathrm{Q}_n^{\mathrm{I}}e_1 + \mathrm{U}_n^{\mathrm{I}}e_2 - \mathrm{V}_n^{\mathrm{I}}e_3\right)_B\}e_3 \end{aligned} \tag{D.3}$$

Stokes quaternions (of incident waves):

for use with Jones matrices,

$$\mathcal{P}_B^T \equiv \mathcal{P}_B^{T+} = \tfrac{1}{\sqrt{2}}\{1 + j(\mathrm{q}\,e_1 + \mathrm{u}\,e_2 + \mathrm{v}\,e_3)_B^T\} \Leftrightarrow \sqrt{2}\,u_B^T\widetilde{u}_B^T * = \frac{1}{\sqrt{2}}\begin{bmatrix} 1+\mathrm{q} & \mathrm{u}-j\mathrm{v} \\ \mathrm{u}+j\mathrm{v} & 1-\mathrm{q} \end{bmatrix}_B^T \tag{D.4}$$

for use with Sinclair matrices,

$$\mathcal{P}_B^{T-} \equiv \mathcal{P}_B^T * = \tfrac{1}{\sqrt{2}}\{1 + j(\mathrm{q}\,e_1 + \mathrm{u}\,e_2 - \mathrm{v}\,e_3)_B^T\} \Leftrightarrow \sqrt{2}\,u_B^T * \widetilde{u}_B^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1+\mathrm{q} & \mathrm{u}+j\mathrm{v} \\ \mathrm{u}-j\mathrm{v} & 1-\mathrm{q} \end{bmatrix}_B^T \tag{D.5}$$

Stokes quaternions of emerging waves are of the type for use with Jones matrices.

APPENDIX E. MUELLER AND KENNAUGH MATRICES – A COMPARISON

In the elliptical basis B of collinear co-phase phasors the two matrices take forms,

Kennaugh matrix (see Appendix A for C_X^B matrix):

$$\mathrm{K}_B = \begin{bmatrix} \mathrm{a}_1 & \mathrm{b}_{1B} & \mathrm{b}_{3B} & \mathrm{b}_{5B} \\ \mathrm{c}_{1B} & \mathrm{a}_{2B} & \mathrm{b}_{4B} & \mathrm{b}_{6B} \\ \mathrm{c}_{3B} & \mathrm{c}_{4B} & \mathrm{a}_{3B} & \mathrm{b}_{2B} \\ \mathrm{c}_{5B} & \mathrm{c}_{6B} & \mathrm{c}_{2B} & \mathrm{a}_{4B} \end{bmatrix} = \widetilde{\mathrm{U}}(A_B \otimes A_B{}^*)\mathrm{U} = \mathrm{D}_X^B \mathrm{K}_X; \quad \mathrm{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -j \\ 0 & 0 & 1 & j \\ 1 & -1 & 0 & 0 \end{bmatrix} \tag{E.1}$$

$$\mathrm{D}_X^B = \widetilde{\mathrm{U}}^* (C_X^B \otimes C_X^B{}^*)\mathrm{U} \tag{E.2}$$

For $A_B = \begin{bmatrix} A_2 & A_3 \\ A_4 & A_1 \end{bmatrix}_B$ and $M_k = A_k A_k{}^*$, k=1,2,3,4, the resulting formulae for elements of the Kennaugh matrix are:

$$\begin{aligned} \mathrm{a}_1 &= \tfrac{1}{2}(M_2 + M_3 + M_4 + M_1)_B & \mathrm{b}_{1B} &= \tfrac{1}{2}(M_2 - M_3 + M_4 - M_1)_B \\ \mathrm{a}_{2B} &= \tfrac{1}{2}(M_2 - M_3 - M_4 + M_1)_B & \mathrm{c}_{1B} &= \tfrac{1}{2}(M_2 + M_3 - M_4 - M_1)_B \end{aligned} \tag{E.3}$$

$$\begin{aligned} \mathrm{a}_{3B} &= \mathrm{Re}(A_4 A_3{}^* + A_2 A_1{}^*)_B \\ \mathrm{a}_{4B} &= \mathrm{Re}(A_4 A_3{}^* - A_2 A_1{}^*)_B \\ \mathrm{b}_{2B} &= \mathrm{Im}(A_2 A_1{}^* + A_4 A_3{}^*)_B \\ \mathrm{c}_{2B} &= \mathrm{Im}(A_2 A_1{}^* - A_4 A_3{}^*)_B \end{aligned} \tag{E.4}$$

$$\begin{aligned} \mathrm{b}_{3B} &= \mathrm{Re}(A_2 A_3{}^* + A_4 A_1{}^*)_B & \mathrm{c}_{3B} &= \mathrm{Re}(A_2 A_4{}^* + A_3 A_1{}^*)_B \\ \mathrm{b}_{4B} &= \mathrm{Re}(A_2 A_3{}^* - A_4 A_1{}^*)_B & \mathrm{c}_{4B} &= \mathrm{Re}(A_2 A_4{}^* - A_3 A_1{}^*)_B \\ \mathrm{b}_{5B} &= \mathrm{Im}(A_2 A_3{}^* + A_4 A_1{}^*)_B & \mathrm{c}_{5B} &= \mathrm{Im}(A_2 A_4{}^* + A_3 A_1{}^*)_B \\ \mathrm{b}_{6B} &= \mathrm{Im}(A_2 A_3{}^* - A_4 A_1{}^*)_B & \mathrm{c}_{6B} &= \mathrm{Im}(A_2 A_4{}^* - A_3 A_1{}^*)_B \end{aligned} \tag{E.5}$$

Mueller matrix (see Appendix A for C_B° matrix):

$$\mathrm{K}_B^{\circ} = \begin{bmatrix} \mathrm{a}_1 & \mathrm{b}_{1B} & \mathrm{b}_{3B} & \mathrm{b}_{5B} \\ \mathrm{c}_{1B}^{\circ} & \mathrm{a}_{2B}^{\circ} & \mathrm{b}_{4B}^{\circ} & \mathrm{b}_{6B}^{\circ} \\ \mathrm{c}_{3B}^{\circ} & \mathrm{c}_{4B}^{\circ} & \mathrm{a}_{3B}^{\circ} & \mathrm{b}_{2B}^{\circ} \\ \mathrm{c}_{5B}^{\circ} & \mathrm{c}_{6B}^{\circ} & \mathrm{c}_{2B}^{\circ} & \mathrm{a}_{4B}^{\circ} \end{bmatrix} = \widetilde{\mathrm{U}}^* (A_B^{\circ} \otimes A_B^{\circ}{}^*)\mathrm{U} = \mathrm{D}_B^{\circ} \mathrm{K}_B \tag{E.6}$$

$$\mathrm{D}_B^{\circ} = \widetilde{\mathrm{U}}(C_B^{\circ} \otimes C_B^{\circ}{}^*)\mathrm{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-2\mathrm{u}^2 & -2\mathrm{u}\,\mathrm{Re}\,\mathrm{w} & -2\mathrm{u}\,\mathrm{Im}\,\mathrm{w} \\ 0 & -2\mathrm{u}\,\mathrm{Re}\,\mathrm{w} & \mathrm{u}^2 - \mathrm{Re}\,\mathrm{w}^2 & -\mathrm{Im}\,\mathrm{w}^2 \\ 0 & -2\mathrm{u}\,\mathrm{Im}\,\mathrm{w} & -\mathrm{Im}\,\mathrm{w}^2 & \mathrm{u}^2 + \mathrm{Re}\,\mathrm{w}^2 \end{bmatrix}_X^B \tag{E.7}$$

Elements of the first raw of Kennaugh and Mueller matrices are identical.

For $A^\circ_B = \begin{bmatrix} A^\circ_2 & A^\circ_3 \\ A^\circ_4 & A^\circ_1 \end{bmatrix}_B = C^\circ_B * A_B$ and $M^\circ_k = A^\circ_k A^\circ_k *$, k=1,2,3,4, the resulting formulae for elements of the Mueller matrix are:

$$\begin{aligned} \mathrm{a}_1 &= \tfrac{1}{2}(M^\circ_2 + M^\circ_3 + M^\circ_4 + M^\circ_1)_B & \mathrm{b}_{1B} &= \tfrac{1}{2}(M^\circ_2 - M^\circ_3 + M^\circ_4 - M^\circ_1)_B \\ \mathrm{a}^\circ_{2B} &= \tfrac{1}{2}(M^\circ_2 - M^\circ_3 - M^\circ_4 + M^\circ_1)_B & \mathrm{c}^\circ_{1B} &= \tfrac{1}{2}(M^\circ_2 + M^\circ_3 - M^\circ_4 - M^\circ_1)_B \end{aligned} \tag{E.8}$$

$$\begin{aligned} \mathrm{a}^\circ_{3B} &= \mathrm{Re}(A^\circ_4 A^\circ_3 * + A^\circ_2 A^\circ_1 *)_B \\ \mathrm{a}^\circ_{4B} &= \mathrm{Re}(A^\circ_4 A^\circ_3 * - A^\circ_2 A^\circ_1 *)_B \\ \mathrm{b}^\circ_{2B} &= \mathrm{Im}(A^\circ_4 A^\circ_3 * + A^\circ_2 A^\circ_1 *)_B \\ \mathrm{c}^\circ_{2B} &= \mathrm{Im}(A^\circ_4 A^\circ_3 * - A^\circ_2 A^\circ_1 *)_B \end{aligned} \tag{E.9}$$

$$\begin{aligned} \mathrm{b}_{3B} &= \mathrm{Re}(A^\circ_2 A^\circ_3 * + A^\circ_4 A^\circ_1 *)_B & \mathrm{c}^\circ_{3B} &= \mathrm{Re}(A^\circ_2 A^\circ_4 * + A^\circ_3 A^\circ_1 *)_B \\ \mathrm{b}^\circ_4 &= \mathrm{Re}(A^\circ_2 A^\circ_3 * - A^\circ_4 A^\circ_1 *)_B & \mathrm{c}^\circ_{4B} &= \mathrm{Re}(A^\circ_2 A^\circ_4 * - A^\circ_3 A^\circ_1 *)_B \\ \mathrm{b}_{5B} &= \mathrm{Im}(A^\circ_2 A^\circ_3 * + A^\circ_4 A^\circ_1 *)_B & \mathrm{c}^\circ_{5B} &= \mathrm{Im}(A^\circ_2 A^\circ_4 * + A^\circ_3 A^\circ_1 *)_B \\ \mathrm{b}^\circ_6 &= \mathrm{Im}(A^\circ_2 A^\circ_3 * - A^\circ_4 A^\circ_1 *)_B & \mathrm{c}^\circ_{6B} &= \mathrm{Im}(A^\circ_2 A^\circ_4 * - A^\circ_3 A^\circ_1 *)_B \end{aligned} \tag{E.10}$$

APPENDIX F. DETERMINATION OF SPECIAL POLARIZATION POINTS FOR SCATTERING MATRIX

Efficient method determines polarization ratios of special incident polarization points (Compare [11]):

$$\rho_B^{T_1,T_2} = \left(\frac{R_1 \mp \sqrt{\Delta}}{R_2}\right)_B^{T_1,T_2} = \tan\gamma_B^{T_1,T_2} \exp\{2\delta_B^{T_1,T_2}\}; \qquad \Delta = R_1{}^2 + R_2 R_3 \tag{F.1}$$

The corresponding Stokes parameters are:

$$\begin{bmatrix} q \\ u \\ v \end{bmatrix}_B^{T_1,T_2} = \begin{bmatrix} \cos 2\gamma \\ \sin 2\gamma \cos 2\delta \\ \sin 2\gamma \sin 2\delta \end{bmatrix}_B^{T_1,T_2} \tag{F.2}$$

Parameters determining special polarization points:

M and N=Mx (incident polarizations for maximum and minimum emergent power)

$$\begin{aligned} R_1 &= \tfrac{1}{2}(A_2 A_2 * - A_3 A_3 * + A_4 A_4 * - A_1 A_1 *)_B = b_{1B} \\ R_2 &= -(A_1 A_4 * + A_3 A_2 *)_B = R_3 * = -b_{3B} + jb_{5B}, \end{aligned} \tag{F.3}$$

O_1 and O_2 (co-pol-nulls),

$$\begin{aligned} R_1 &= -A_{3B} - A_{4B} \\ R_2 &= 2A_{1B}, \quad R_3 = -2A_{2B}, \end{aligned} \tag{F.4}$$

E_1 and E_2 (cross-pol-nulls or eigenpolarizations); they exist if $\Delta \geq 0$,

$$\begin{aligned} R_1 &= (A_1 A_1 * - A_2 A_2 * + A_3 A_4 * - A_4 A_3 *)_B \\ R_2 &= 2(A_1 A_3 * + A_3 A_2 *)_B \\ R_3 &= 2(A_2 A_4 * + A_4 A_1 *)_B \end{aligned} \tag{F.5}$$

K and L=Kx (co-pol-max/saddle points or characteristic polarizations determining the Q_K axis),

$$\begin{aligned} R_1 &= (A_2 A_2 * - A_1 A_1 *)_B \\ R_2 &= -A_{1B}(A_3 * + A_4 *)_B - A_{2B} * (A_3 + A_4)_B = R_3 *. \end{aligned} \tag{F.6}$$

In the case of polarizations M and N simpler formula can be more convenient directly expressing their normalized Stokes parameters:

$$\begin{bmatrix} q \\ u \\ v \end{bmatrix}_B^{M,N} = \frac{\pm 1}{b_0} \begin{bmatrix} b_1 \\ b_3 \\ b_5 \end{bmatrix}_B ; \quad b_0 = \sqrt{b_{1B}^2 + b_{3B}^2 + b_{5B}^2} \tag{F.7}$$

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